0-ALGEBRA TOOL BOX

In this lecture, we'll cover some of the common skills needed to be successful in an algebra course. These skills are as follows:

- Simplifying expressions
- Solving equations
- Dealing with exponents
- Dealing with roots and radicals

Simplifying Expressions

In algebra, an expression is simply a mathematical statement that doesn't equal anything yet. The following statements are algebraic expressions:

$$x + 3$$
$$x + 3y + 5$$
$$2x - 3y + 6z - 11$$

We begin our toolbox by taking a look at the steps needed to simplify an expression. Basically, this means getting the expression into the simplest form possible. Let's simply the following expression:

$$2x + 5x - 11$$

In this expression, there are two x-terms. We call them x-terms because they both 2x and 5x include the variable x in them. The numbers in front of the x are referred to as coefficients. By the way, when you see something like 2x, it means that 2 is multiplied by x. To simplify this expression, we would add 2x and 5x to get 7x. Our expression becomes …

$$7x - 11$$

And that's it, we've simplified 2x + 5x - 11 to get 7x - 11. Notice that we did not write this like 2x + 5x - 11 = 7x - 11 because we are only dealing with expressions and not equations at this point. Let's do another one with more moving parts. Simplify the following expression:

$$3x^2y + 7xy^2 + 6x^2y - 4xy^2 - 5xy$$

In this expression, we have some x^2y terms, some xy^2 terms and an xy term. Let's rearrange things and color-code the *like terms* in this expression.

$$3x^2y + 6x^2y + 7xy^2 - 4xy^2 - 5xy$$

Now we can perform the indicated operations involving the like terms in this expression to get the following:

$$9x^2y + 3xy^2 - 5xy$$

Therefore, $3x^2y + 7xy^2 + 6x^2y - 4xy^2 - 5xy$ simplified becomes $9x^2y + 3xy^2 - 5xy$.

There are times when it is necessary to move some parentheses out of the way to simplify an expression.

Consider how you might simplify the following: $(3x - y + 1) + (5x + 2y - 3)$

First, we'll drop the parentheses: $3x - y + 1 + 5x + 2y - 3$

Now, collect the like terms (they've been colored): $3x + 5x - y + 2y + 1 - 3$

Perform the operations on the like terms: $8x + y - 2$

Therefore, the expression $(3x - y + 1) + (5x + 2y - 3)$ simplified becomes $8x + y - 2$.

When there is a number sitting just outside of a parentheses, it means that the number is to be multiplied by everything in that parentheses.

Simplify the following expression: $-3(x - 2y)$

Multiply -3 by everything within the parentheses: $-3(x) + (-3)(-2y)$

Complete the operations $-3x + 6y$

Therefore, the expression $-3(x - 2y)$ simplified becomes $-3x + 6y$.

This process uses the **distributive property** which states that $a(b + c) = ab + ac$.

Simply the following expression: $2(3x + 5y) - (x + 2y)$

Distribute the 2 and -1 as required: $2(3x) + 2(5y) - (x) - (2y)$

Perform the necessary multiplication: $6x + 10y - x - 2y$

Collect like terms: $6x - x + 10y - 2y$

Perform the necessary operations: $6x + 8y$

Therefore, the expression $2(3x + 5y) - (x + 2y)$ simplified becomes $6x + 8y$.

Solving Equations

Algebra has the well earned reputation for being the course where you learn how to solve for x. The variable x would normally be a component of some kind of equation where two expressions equal each other. There are only four properties of equations that you need to know to learn how to solve them.

1. **Addition Property** - Adding the same number to both sides of an equation gives an equivalent equation. For example, $x - 5 = 2$ is the same as $x - 5 + 5 = 2 + 5$ or $x = 7$
2. **Subtraction Property** - Subtracting the same number to both sides of an equation gives an equivalent equation. For example, $z + 15 = -2$ is the same as $z + 15 - 15 = -2 - 15$ or $z = -17$
3. **Multiplication Property** - Multiplying both sides of an equation by the same number gives an equivalent equation. For example, $y/6 = 5$ is the same as $6(y/6) = 6(5)$ or $y = 30$
4. **Division Property** - Dividing both sides of an equation by the same number gives an equivalent equation. For example, $7x = -49$ is the same as $7x/7 = -49/7$ or $x = -7$

Let’s put these rules to the test.

Solve for x in the following equation:	$3x - 2 = 4 - 7x$
Add 2 to both sides:	$3x - 2 + 2 = 4 - 7x + 2$
This gives:	$3x = 6 - 7x$
Now, add 7x to both sides:	$3x + 7x = 6 - 7x + 7x$
This gives:	$10x = 6$
Divide both sides by 10:	$10x/10 = 6/10$
Now, we have our solution:	$x = 6/10 = 0.6$

Let’s try another one with a lot more moving parts!

Solve for x in the following equation:	$3(4x - 3) = 2(6 + x)$
Apply the distributive property:	$3(4x) - 3(3) = 2(6) + 2x$
Perform the required operations:	$12x - 9 = 12 + 2x$
Add 9 to both sides:	$12x - 9 + 9 = 12 + 2x + 9$
Perform required operations:	$12x = 21 + 2x$
Subtract 2x from both sides:	$12x - 2x = 21 + 2x - 2x$

Perform required operations: $10x = 21$

Divide both sides by 10: $10x/10 = 21/10$

Perform the final operation to get the solution: $x = 21/10 = 2.1$

Dealing with Exponents

An exponent is the number that represents the power to which you've raised a specific number or variable. For example, in 2^2 the 2 is the exponent (i.e. we are raising 2 to the power of 2). In x^3, 3 is the exponent (i.e. we are raising x to the power of 3). When simplifying expressions, or solving equations, it will occasionally be necessary to use one or more of the following properties of exponents:

For all real numbers a and b and positive integers, m and n, the following apply:

Law of Product: $a^m a^n = a^{m+n}$

Example: $a^5 a^2 = a^{5+2} = a^7$

Law of Quotient: $\frac{a^m}{a^n} = a^{m-n}$

Example: $\frac{a^5}{a^2} = a^{5-2} = a^3$

Law of Zero Exponent: $a^0 = 1$

Example: $5^0 = 1$

Law of Negative Exponent: $a^{-m} = \frac{1}{a^m}$

Example: $a^{-3} = \frac{1}{a^3}$

Law of Power of a Power: $(a^m)^n = a^{mn}$

Example: $(a^5)^2 = a^{5\cdot 2} = a^{10}$

Law of Power of a Product: $(ab)^m = a^m b^m$

Example: $(ab)^3 = a^3 b^3$

Law of Power of a Quotient: $(\frac{a}{b})^m = \frac{a^m}{b^m}$

Example: $(\frac{a}{b})^3 = \frac{a^3}{b^3}$

Let's work some examples:

Evaluate: $\frac{5^6}{5^4}$

Solution: $\frac{5^6}{5^4} = 5^{6-4} = 5^2 = 25$

Simplify: $\frac{y^2}{y^5}$

Solution: $\frac{y^2}{y^5} = y^{2-5} = y^{-3} = \frac{1}{y^3}$

Simplify: $(3xy)^3$

Solution: $(3xy)^3 = 3^3x^3y^3 = 27x^3y^3$

Simplify: $3^{15-2m}3^{2m}$

Solution: $3^{15-2m}3^{2m} = 3^{15-2m+2m} = 3^{15}$

Dealing with roots and radicals

If you have a square with each side length *a*, then the area of the square is $a \cdot a = a^2$ units. Therefore the length of any side of the square can be represented as the *square root* of a^2 shown as follows: $\sqrt{a^2} = a$. Generally, we can represent the *n**th root* of some number *a* as $\sqrt[n]{a}$ where the symbol $\sqrt{\ }$ is called the *radical* and *n* is the *index of the radical*, and *a* is the *radicand*. Here are the properties of roots and radicals:

- If *a* is a positive number and *n* is even, there are *two real* n^{th} *roots* of a, $\sqrt[n]{a}$ and -$\sqrt[n]{a}$.
  - Example: $\sqrt[4]{81} = 3$ because $3^4 = 81$
- If *a* is a negative number and *n* is even, there is *no real* n^{th} *root* of *a*.
  - Example: $\sqrt[4]{-81} = undefined$ because $3^4 = 81$ and $(-3)^4 = 81$
  - Not a real number

- If *a* is a real number and *n* is odd, then $\sqrt[n]{a^n} = a$.
  - Example: $\sqrt[3]{(-2)^3} = -2$ or $\sqrt[3]{-8} = -2$

- If *a* is a real number and *n* is even, then $\sqrt[n]{a^n} = |a|$.
  - Example: $\sqrt[10]{2^{10}} = 2$

- If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$
  - Example: $\sqrt[3]{8} \cdot \sqrt[3]{8} = \sqrt[3]{8 \cdot 8} = \sqrt[3]{64} = 4$

- If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$
  - Example: $\frac{\sqrt[3]{64}}{\sqrt[3]{8}} = \sqrt[3]{\frac{64}{8}} = \sqrt[3]{8} = 2$

It is also possible to represent expressions involving radicals as expressions with fractional exponents.

Here's the general rule for rational exponents: $\sqrt[n]{a} = a^{\frac{1}{n}}$

For example: $a^{\frac{1}{2}} = \sqrt{a}$ so it follows that $(a^{\frac{1}{2}})^2 = a$ for $a \geq 0$

Example: $\sqrt[3]{27} = 27^{\frac{1}{3}} = 3$

Write the following expressions with exponents rather an radicals:

$\sqrt[3]{x^2} = x^{\frac{2}{3}}$

$\sqrt{(3xy)^5} = (3xy)^{\frac{5}{2}}$

Simplify the expression: $\sqrt[3]{x^6y^5}$

Solution: $\sqrt[3]{x^6y^5} = (x^6y^5)^{\frac{1}{3}} = x^{\frac{6}{3}}y^{\frac{5}{3}} = x^2y^{\frac{3}{3}}y^{\frac{2}{3}} = x^2y\sqrt[3]{y^2}$

Inequalities

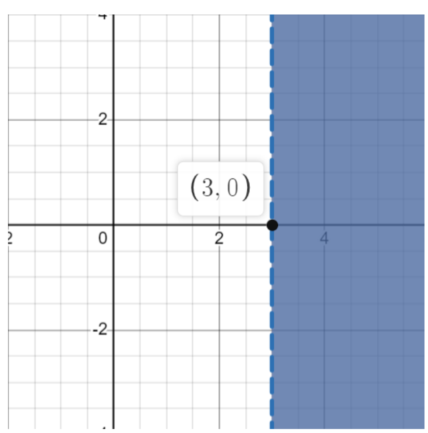
Up till this point, we've been dealing primarily with linear equations. In general, a linear equation is a function where we can express one variable in terms of another such as the equation of a line: $y = mx + b$. In this case, there is one unique solution y for every value of x. Now, let's briefly explore the subject of inequalities. As it sounds, an inequality is a function where there is a conditional relationship that must be considered when finding the solution. The typical conditional relationships are as follows:

Greater than	$>$
Less than	$<$
Greater than or equal to	$\geq$
Less than or equal to	$\leq$

The statement $4x + 3 < 7x - 6$ is a linear inequality where the value of x that satisfies the inequality forms a solution set for the inequality. To solve an inequality, you can employ the same algebraic tools you would to solve a linear equation. For example…

$$4x + 3 < 7x - 6$$
$$4x - 7x < -6 - 3$$
$$-3x < -9$$
$$x > 3$$

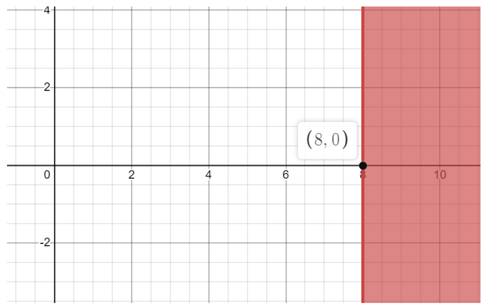
So, the solution set to $4x + 3 < 7x - 6$ is $x > 3$. Notice that when we divide by -3 on both sides, it requires that we flip the inequality sign from < to >. Below, we see the graph of $4x + 3 < 7x - 6$ showing that the solution is all x values to the right of the point (3, 0). We can represent this solution as x > 3 or in interval notation as (3, ∞) meaning that x can be anything greater than 3.



Let's consider another example.

$$7x \geq 8 + 6x$$
$$7x - 6x \geq 8$$
$$x \geq 8$$

The solution set for $7x \geq 8 + 6x$ is $x \geq 8$ which can also be represented in interval notation as $[8, \infty)$. In this format, the bracket [implies that x can equal 8 and any value greater than 8. This is shown in the graph below.



The following table provides a comparison between the symbols used in inequality notation and interval notation.

Interval Notation	*Inequality Notation*	*Meaning*
(a, ∞)	$x > a$	x is greater than a
$[a, \infty)$	$x \geq a$	x is greater than or equal to a
$(-\infty, b)$	$x < b$	x is less than b
$(-\infty, b]$	$x \leq b$	x is less than or equal to b
(a, b)	$a < x < b$	x is between a and b
$[a, b)$	$a \leq x < b$	x can equal a and is between a and b
$(a, b]$	$a < x \leq b$	x is between a and b and can equal b
$[a, b]$	$a \leq x \leq b$	x is between a and b and includes a and b

Example Problem: A cat's health is at risk when the body temperature is above 104°F or higher. What temperature reading in Celsius would indicate that our cat's health is at risk? Recall that the $F = \frac{9}{5}C + 32$ is the formula that relates Fahrenheit to Celsius.

Solution: The problem states that a cat's health is at risk when $F \geq 104$. Substituting this value into our temperature formula above, we get the following:

$$\frac{9}{5}C + 32 \geq 104$$

Solving for C, we get the following:

$$\frac{9}{5}C + 32 \geq 104$$
$$\frac{9}{5}C \geq 104 - 32$$
$$\frac{9}{5}C \geq 72$$
$$5 \times \frac{9}{5}C \geq 5 \times 72$$
$$9C \geq 360$$
$$C \geq \frac{360}{9}$$
$$C \geq 40$$

Therefore, a cat's health is at risk when body temperature is at or above 40ºC.

Example Problem: During the summer of 2011, Dallas, Texas experienced 40 consecutive days where the temperature was at least 108ºF. On many of these days, the combination of humidity and temperature made it feel hotter than the actual air temperature. When the temperature is 100ºF, the heat index A depends upon the percentage humidity h (expressed as a decimal) by the following formula:

$$A = 90.2 + 41.3h$$

What would be the range of humidity levels that would result in a heat index of at least 108ºF?

Solution: If the heat index is at least 108ºF, the inequality we need to solve would be as follows:

$$A \geq 108$$
$$90.2 + 41.3h \geq 108$$
$$41.3h \geq 108 - 90.2$$
$$41.3h \geq 108 - 90.2$$
$$41.3h \geq 17.8$$
$$h \geq 17.8/41.3$$
$$h \geq 0.431$$

Therefore, humidity levels greater than or equal to 43.1% have the potential to result in a heat index of 108ºF. Since we can not have more than 100% humidity, that means humidity levels between 43.1% and 100% can produce heat indices of 108ºF. This result can also be expressed as a double inequality like the following: $0.431 \leq h \leq 1.00$