5- EXPONENTIAL AND LOGARITHMIC FUNCTIONS

An exponential function is any mathematical function that can be expressed in the following form:

$$f(x) = ab^x$$

In this format, a is a constant and is typically the initial value of the function when $x = 0$. Recall that anything raised to the power of zero equals 1, so $f(0) = ab^0 = a(1) = a$. The value b is the base, and it is always greater than zero (*b > 0*). The variable is *x* and it represents the exponent of the function.

Example: Evaluate $f(x) = 2(10)^x$ when *x = 2*.

Solution: $f(2) = 2(10)^2 = 2(100) = 200$

Example: Evaluate $f(x) = 2(10)^{(x-1)}$ when *x = 2*.

Solution: $f(2) = 2(10)^{(2-1)} = 2(10)^1 = 2(10) = 20$

Exponential Growth

An exponential function can be used to model growth for a wide range of phenomena. In an exponential growth function, the base *(1+r)* is always greater than 1 and *a > 0*. Exponential growth functions follow this format:

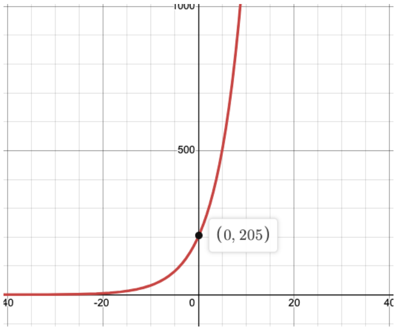
$$f(x) = a(1 + r)^x$$

In this formula, *r* represents the rate of growth.

Example: In 2020, the population of rabbits on Hop Island was estimated to be 205. If the annual rabbit population grows 20%, estimate the rabbit population on the island in 2030.

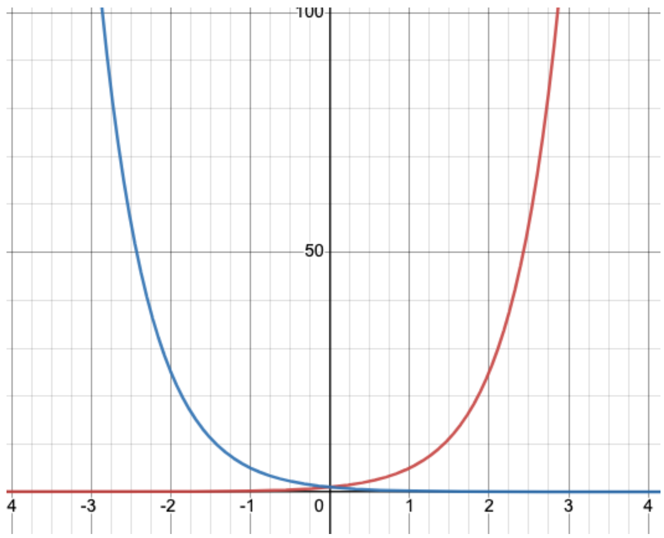
Solution: Our beginning population is $a = 205$, and our growth rate is 20% which will need to be expressed as a decimal in the formula so *r = 0.2*. Additionally, we are estimating the population after 10 years, so *x = 10*.

$$f(10) = 205(1 + .2)^{10} = 205(1.2)^{10} = 1269.3$$



We'll go ahead and round this result to 1269 because we don't want partial rabbits. Therefore, our estimated rabbit population on Hop Island in 2030 will be 1269.

The graph shows what $f(x) = 205(1.2)^x$ looks like. Notice that the y-intercept is the point (0, 205) and that it is asymptotic with the x-axis.



The red line represents the function $f(x) = 5^x$ and the blue line represents the function $f(x) = 5^{-x}$.

Notice that the blue line is the reflection of the red line about the y-axis.

The red line represents an exponential growth function. The blue represents an exponential decay function.

Exponential Decay

An exponential function can be used to model decay for a wide range of phenomena. Exponential decay functions follow this format:

$$f(x) = a(1 - r)^x$$

In this exponential decay function, the base *(1-r)* is always less than 1, $a > 0$, and r represents the rate of decay.

Another option is for the exponent to be negative. In that case, the exponential decay function will follow this format:

$$f(x) = ab^{-x}$$

In this version of the exponential decay function, b > 1 and $a > 0$.

Example: An adult Saguaro cactus can store up to 1500 gallons of water. During extreme dry periods, the Saguaro can lose up to ⅔ of this stored water and still survive. The rate at which stored water is used depends upon a range of environmental conditions. If a particular adult Saguaro cactus uses 5% of its stored water monthly during an extremely long hot drought, estimate the amount of stored water that will remain after 6 months?

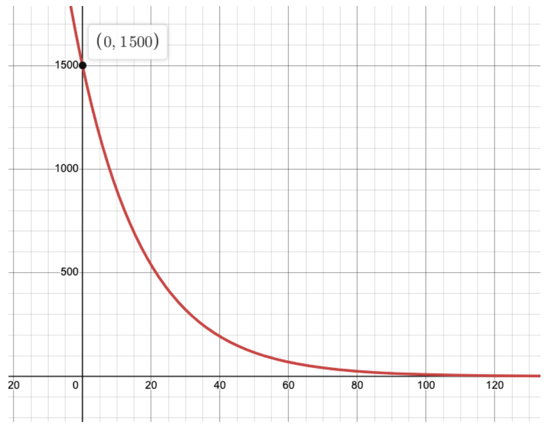
Solution: We need to set up our exponential decay function with the provided parameters. In this case, we know that our initial amount of water is 1500 gallons, so $a = 1500$. The rate at which water is used will be $r = 0.05$. Here's our exponential decay function:

$$f(x) = 1500(1 - 0.05)^x = 1500(0.95)^x$$

Now we are ready to estimate how much water will remain stored in the cactus after 6 months. To do this we evaluate *f(6)*.

$$f(6) = 1500(0.95)^6 = 1102.64\ gallons$$

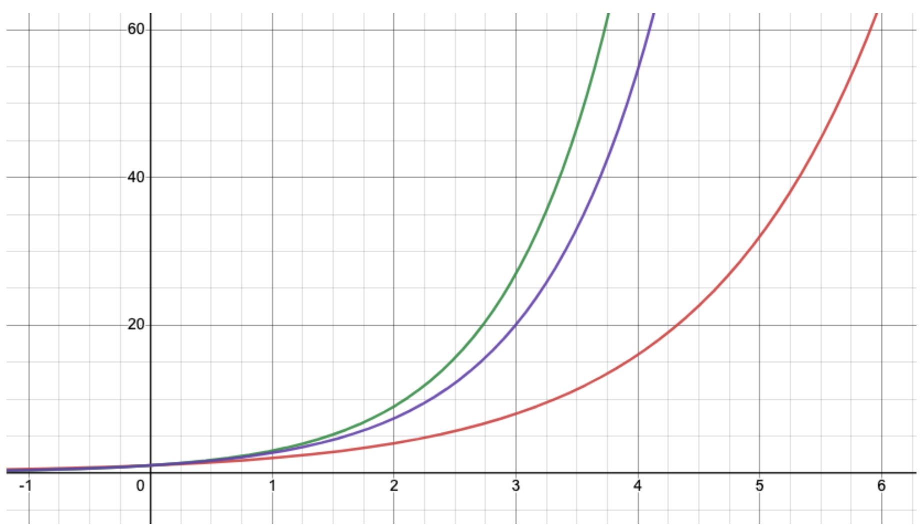
Here's the graph of our exponential decay function $f(x) = 1500(0.95)^x$ which models water usage within an adult Saguaro cactus.



The Number *e* (Euler's Number)
There are many application areas in the sciences and business that are modeled by exponential functions with the base *e*. The number *e* is an irrational number that approximately equals:

$$e = 2.718281828...$$

In the graph below, the red line represents $f(x)=2^x$ and the green line represents $f(x)=3^x$. The graph of $f(x)=e^x$ is between these functions and is represented by the purple line.

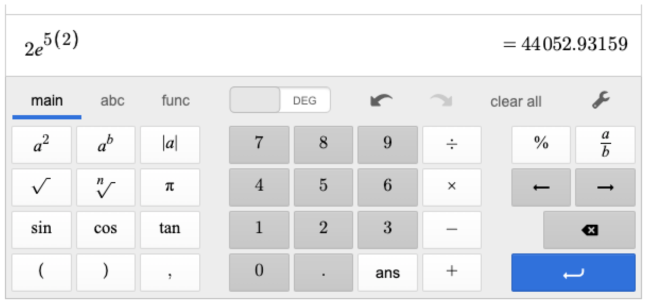


A typical exponential function base *e* will follow this format: $f(x) = e^{rx}$
- If $r > 1$, then the function base *e* will represent exponential growth.
- If $0 < r < 1$, then the function base *e* will represent exponential decay.

Example: Evaluate the function $f(x) = 2e^{5x}$ when $x = 2$.

Solution: $f(x) = 2e^{5x} = 2e^{5(2)} = 2e^{10} = 44052.93$

It is highly recommended that you use a scientific calculator for these computations. Here's what this solution would look like using https://www.desmos.com/scientific.



The typical format for an exponential decay formula base e will be $f(x) = ae^{-rx}$

For example, an exponential decay function is used to model the amount of radioactive material remaining after a period of time. Carbon-14 decays over time with the amount remaining after t years represented by the following function:

$$f(t) = y_o e^{-0.000121t}$$

Where y_o is the initial amount of carbon-14 material and $r = 0.000121$ is the rate of change per year.

Example: If the original amount of carbon-14 present in an artifact is 100 grams, how much remains after 2000 years?

Solution: $f(2000) = 100e^{-0.000121(2000)} = 78.51\ grams$

Here's the solution using https://www.desmos.com/scientific.

Let's do another one.

Example: What percentage of the original amount of carbon-14 remains after 4500 years?

Solution: We know that y_o is the initial amount of carbon-14 material, so plugging t = 4500 into our formula gives $f(4500) = y_o e^{-0.000121(4500)} = 0.57y_o$.

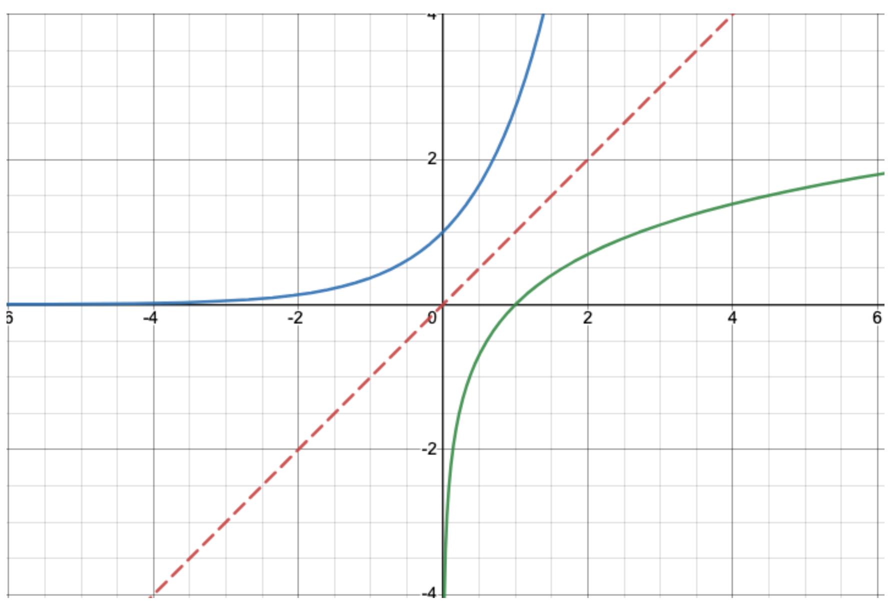
Therefore, 57% of the carbon-14 will remain after 4500 years.

Incidentally, the half-life for carbon-14 is 5730 years which means that after this time only 50% of the original carbon-14 remains.

Logarithmic Functions

Logarithmic functions have the following format: $f(x) = log_b x$

Logarithmic functions are the inverse of exponential functions. In general, a function is represented as *y = f(x)* and its inverse function is represented as *x = f(y)*. It is important to note that a function and its inverse function must have a one-to-one correspondence between the independent and dependent variables defining the function. The inverse function for a logarithmic function is an exponential function. This is shown graphically below:



The green line is the graph of *f(x) = ln x*. This function is called the *natural log of x* and it is base *e*.

The blue line is the function *f(x) = e^x^* which is the inverse function for *f(x) = ln x*.

Since *f(x) = ln x* and *f(x) = e^x^* are inverses of each other, their graphs are symmetrical about the line *y = x* represented by the red dotted line.

Therefore, since logarithmic functions have the format $y = log_b x$, the corresponding exponential form will be $x = b^y$, and its inverse will have the format $y = b^x$.

These are referred to as the logarithmic forms and exponential forms respectively as demonstrated by the examples in the following table:

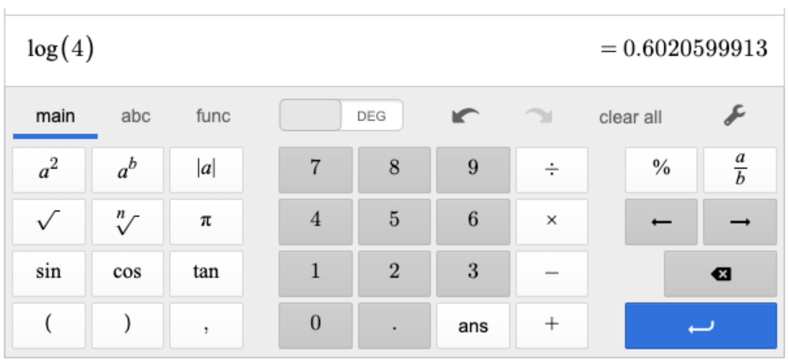
Logarithmic Form	Exponential Form
$y = \log_2 x$	$x = 2^y$
$2 = \log_{10}100$	$10^2 = 100$
$2 = \log_3 9$	$3^2 = 9$

Example: Evaluate the function $f(x) = log_{10}\, x$ when x = 4.

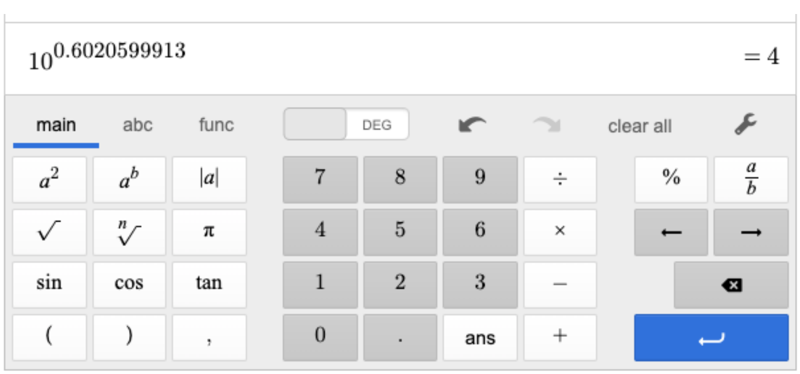
To find *f(4)*, we can input this into a calculator to quickly find the solution. Scientific calculators come with a log function button that is base 10 by default.

$f(4) = \log_{10}4 = 0.6020599913$

Here's the solution using https://www.desmos.com/scientific. The log button is found on the **func** menu.



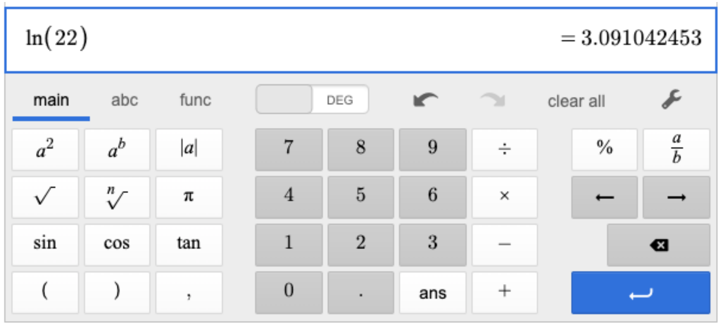
So, our logarithmic form works out to be $\log_{10}4 = 0.6020599913$. The exponential form for this would be $10^{0.6020599913} = 4$ which we can verify with the calculator as shown below:



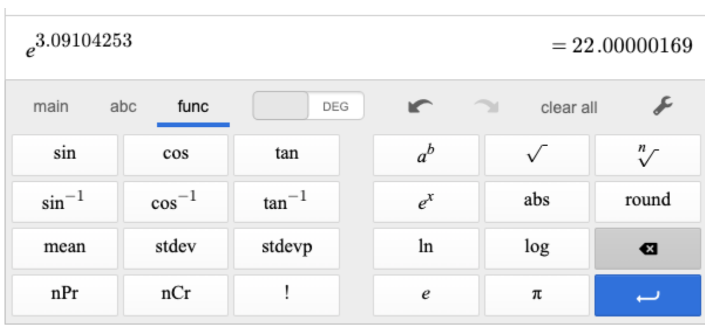
Example: Evaluate the function $f(x) = \ln x$ when x = 22.

Another common logarithm is the *natural logarithm* which is a logarithm based *e*. This is defined as follows: $\log_e x = \ln x$. It is handy to know that all scientific calculators come with an *ln* function button.

So, let's use https://www.desmos.com/scientific to evaluate $f(22) = \ln 22$. Here's the solution:



Therefore, $f(22) = \ln 22 = 3.09104253$. The exponential form would be $e^{3.09104253} = 22$ as verified below:



An interesting property of logarithmic functions is the following:

$$f(x) = \log x^n = n \log x$$

This is called the **power property** of logarithms. Let's put this property to use in the following example:

Example: Evaluate the function $f(x) = \log x^4$ when $x = 10$. Round your answer to the nearest hundredths.

Solution: $f(10) = \log 10^4 = 4 \log 10 = 4(1) = 4$

This solution employed the following basic property of logarithms: $\log_b b = 1$

The following list contains some of the basic logarithmic properties (for b > 0, b ≠ 1):

- $log_b b = 1$
- $log_b 1 = 0$
- $log_b b^x = x$
- $b^{\log x} = x$

The following list contains some useful operational properties of logarithms (for b > 0, b ≠ 1).

- Product Property: $log (MN) = log M + log N$
- Quotient Property: $log (\frac{M}{N}) = log M - log N$
- Product Property: $log M^n = n log M$

A Logarithmic Function Application: pH

The acidity or basicity of a solution depends upon the concentration of free hydrogen ions [H+] in the solution. The pH scale is a logarithmic scale that represents a simple way to represent the hydrogen ion concentration of a solution. The pH of a solution is given by the following formula:

$$pH = -\log[H+]$$

Example: Find the pH value of beer for which the hydrogen concentration is [H+] = 0.0000631.

Solution: pH = -log(0.0000631) = 4.199970641. Therefore, the pH of this beer is 4.2.

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Example: The pH value of eggs is 7.79. Find the hydrogen ion concentration for eggs.

Solution: Since the logarithmic form is 7.79 = -log[H+], it follows that the exponential form will give the solution [H+] = $10^{-7.79}$ = 0.00000001628101