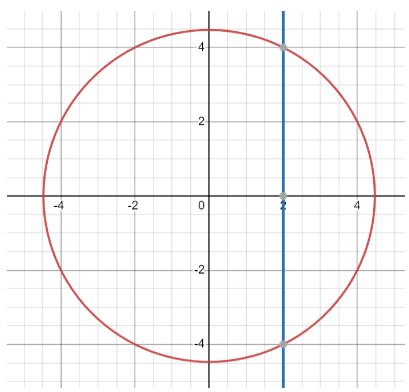
3-FUNCTIONS

A function is an algebraic equation that relates one variable (the input or independent variable) to another variable (the output or dependent variable). So far, we've dealt with the equation of a line using the format $y = mx + b$. We can express this in function notation as $f(x) = mx + b$.

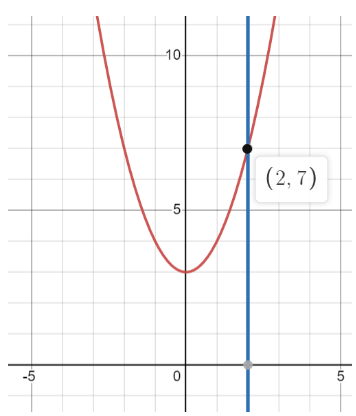
For example, let's say we have the equation of a line given by the formula $y = 3x - 5$. In functional notation, this line equation will be represented as $f(x) = 3x - 5$. Let's evaluate this function when $x = 2$.

$$f(2) = 3(2) - 5 = 6 - 5 = 1$$

Therefore, the process of evaluating a function is a matter of inputting the value for x and performing required operations to determine the output value $f(x)$. Please note that, by definition, a function has only one output for each input value. As such, a function must pass the **vertical line test**. In this test, a vertical line drawn through the function can only intersect the graph of the function at only one point. If it intersects at more than one point, the relationship fails the vertical line test and is not a function. For example, the graph of the circle $x^2 + y^2 = 20$ below is not a function because the vertical line $x = 2$ intersects at two points.

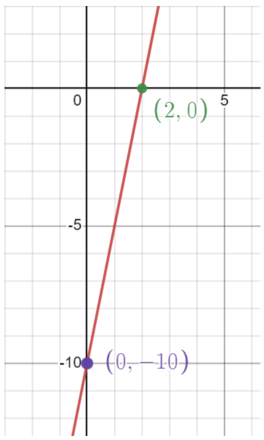


The graph of the parabola $f(x) = x^2 + 3$ only intersects the line $x=2$ at one point, so it is a function because it passes the vertical line test.



The set of inputs for function *f(x)* is referred to as the **domain** of the function. The set of all outputs for function *f(x)* is referred to as the **range** of the function.

Example Problem: What is the domain and range for the function $f(x) = 5x - 10$?



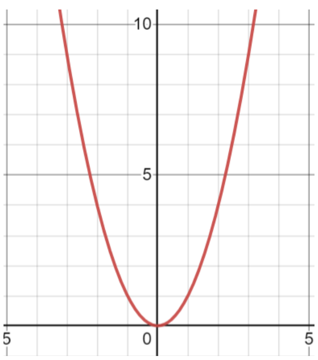
Solution: The image shows that $f(x) = 5x - 10$ is a line with intercepts *(2, 0)* and *(0, -10)*.

Since we can plug in any value of x and get a value for f(x), this means that the domain of this function is the set of all real numbers. We can indicate the domain in interval notation as $(-\infty, \infty)$.

Similarly, the range is also the set of all real numbers and can also be represented as $(-\infty, \infty)$.

Power Functions

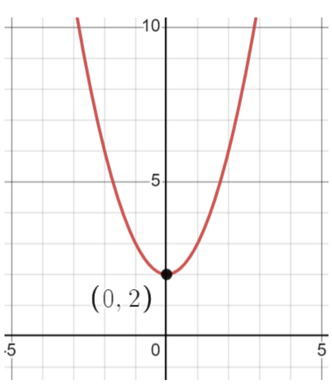
As the name implies, power functions are functions where the input variable is raised to a power. These functions are of the form $f(x) = x^n$. The graph of power functions usually results in some kind of curve. For example, the graph of $f(x) = x^2$ is a parabola that passes through the point (0, 0).



The domain of this function is $(-\infty, \infty)$ because we can input any *x* and get a value for *f(x)*.

However, the range of this function is restricted to all values of f(x) greater than 0. So, we would represent the range for $f(x) = x^2$ as $[0, \infty)$.

For this parabola, the **vertex** is (0, 0).



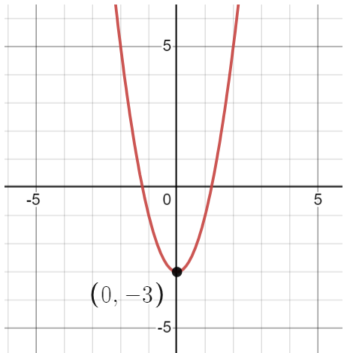
Here we have the graph of $f(x) = x^2 + 2$.

Like before, the domain of this function is $(-\infty, \infty)$.

Notice that the vertex of the parabola is now (0, 2). This means that the range of the function is any value of *f(x)* that is 2 or higher. As such, the range is represented as $[2, \infty)$.

To find the vertex of a function, you simply set $x = 0$ and find the value of the y-intercept.

Example Problem: Consider the function $f(x) = 2x^2 - 3$. What is the vertex of this function?



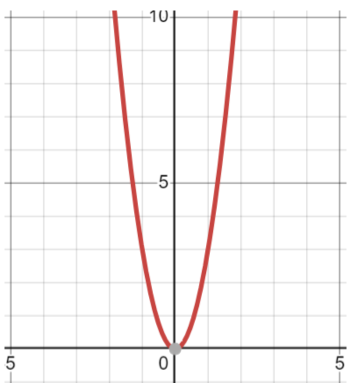
Solution: Set x = 0. $f(0) = 2(0)^2 - 3 = -3$.

Therefore, the vertex for this parabola is the point (0, -3) as shown in the graphic.

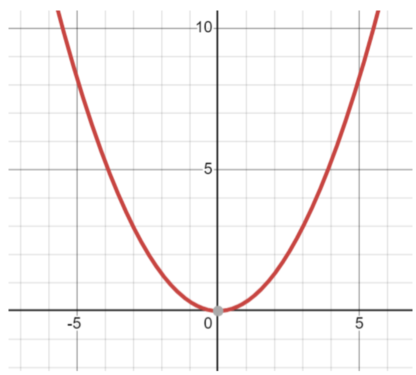
The domain is (-∞, ∞).

The range is [-3, ∞).

When it comes to the formula for a parabola, the coefficient on the x^2 term determines the width of the parabola as the following examples show:

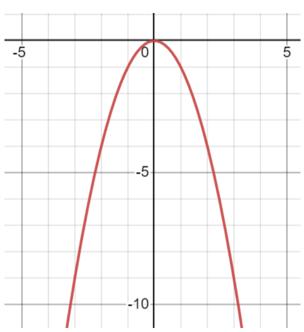


$f(x) = 3x^2$



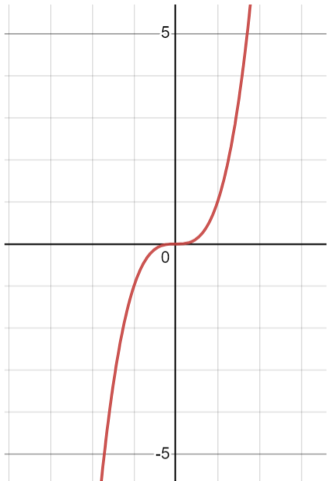
$f(x)=0.33x^2$

The sign on the coefficient determines the direction that the parabola opens. A positive coefficient means that the parabola will open upwards. A negative coefficient means the parabola opens downward.



$f(x) = -x^2$

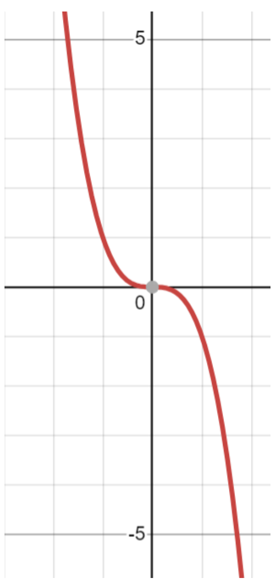
Here we see the graph of the function $f(x) = x^3$



The domain of this function is $(-\infty, \infty)$.

The range of this function is $(-\infty, \infty)$.

The sign on the coefficient of the x^3 term determines the orientation of this cubic function. A negative coefficient flips the curve around the y-axis like so:

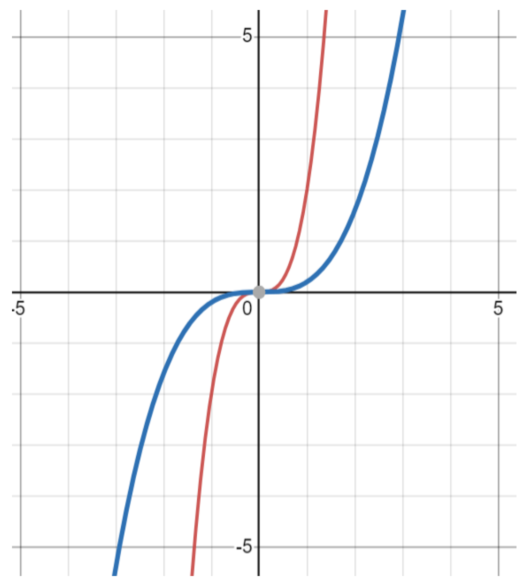


This is the function $f(x) = -x^3$

The value of the coefficient determines the width of the curve.

In the graph to the right, the red line is the function $f(x) = 2x^3$ and the blue line is the function $f(x) = 0.2x^3$.

The domain and range for both of these functions is $(-\infty, \infty)$.



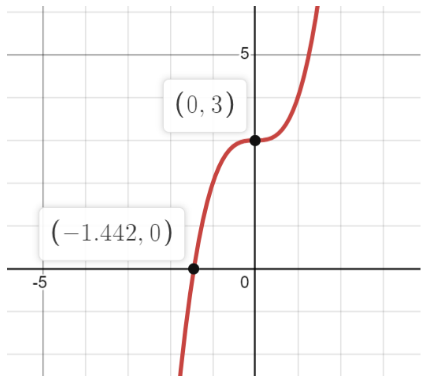
Example Problem: Find the intercepts and graph the function $f(x) = x^3 + 3$.

Solution: Letting $x = 0$, we find that $f(0) = (0)^3 + 3 = 3$ so the y-intercept is the point (0, 3).

Setting $f(x) = 0$, we find that $0 = x^3 + 3$. This means that $x^3 = -3$. To find the value of x that satisfies this, we need to take the cube root of both sides. Recall that the online calculator https://www.desmos.com/scientific provides an easy way to find these kinds of solutions.

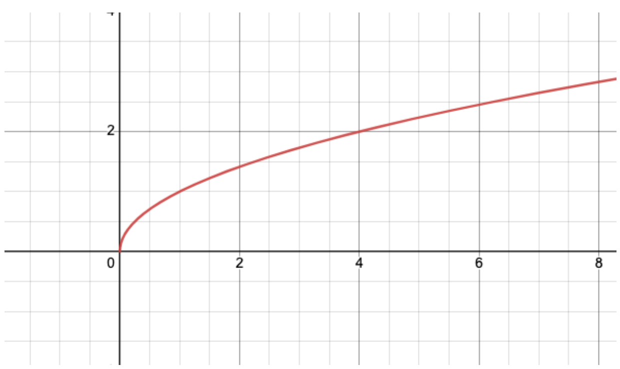
$$\sqrt[3]{x^3} = \sqrt[3]{-3}$$
$$x = -1.442$$

This means that the point (-1.442, 0) will be the x-intercept for the function *f(x)* = $x^3 + 3$. The graph below shows the curve *f(x)* = $x^3 + 3$ with intercepts (-1.442, 0) and (0, 3).



Radical Functions

Radical functions are functions that involve a radical sign and follow the form $f(x) = \sqrt[n]{x}$. You may recognize the radical sign $\sqrt{\ }$ as the square root sign. However, this symbol is more properly represented as $\sqrt[n]{\ }$ where n represents the root (*a.k.a. n^{th} root*). So, $\sqrt[2]{\ }$ would be our old friend the square root radical and $\sqrt[3]{\ }$ would be the cube root radical.



This is the graph of $f(x) = \sqrt{x}$

Since we are taking the square root here, we are limited to only positive values of x. Similarly, our output will only result in positive values for y. As such, the domain and range would be:

Domain: [0, ∞)
Range: [0, ∞)

By employing what are called **imaginary numbers**, we can come up with solutions that involve negative numbers under the radical. To do this, Rene Descartes (17th century) defined $i = \sqrt{-1}$. This tool provided a clever way to evaluate radical functions. At the time, the concept of *i* was considered to be fictitious, hence the name imaginary number. However, very quickly a wide range of applications in the natural sciences and business were identified that employed this approach. Regardless, we are still stuck with calling them imaginary numbers.

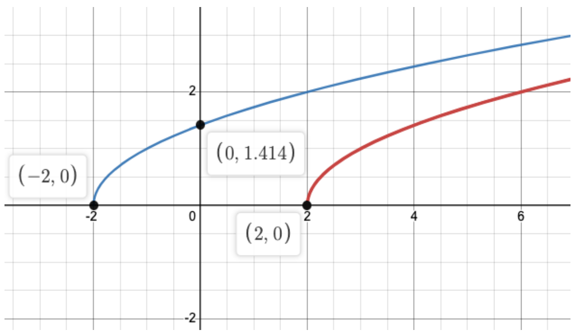
Imaginary Number Properties and Examples:

$i = \sqrt{-1}$ $\quad i^2 = (\sqrt{-1})^2 = -1$ $\quad i^3 = (\sqrt{-1})^3 = (\sqrt{-1})^2\sqrt{-1} = (-1)i = -i$

$(5i)^2 = 25i^3 = 25(-i) = -25$ $\quad (3i)^4 = 81i^4 = 81i^2i^2 = 81(-1)(-1) = 81$

Imaginary numbers will make an appearance in the next unit, so we'll pause our conversation about them until then.

In radical functions for the form $f(x) = \sqrt{x + a}$, the a represents that x-intercept. For example, in the graph below, the red line represents the function $f(x) = \sqrt{x - 2}$ with an x-intercept of (-2, 0). The blue line is the function $f(x) = \sqrt{x + 2}$ with an x-intercept of (2, 0). Note also that $f(x) = \sqrt{x + 2}$ has a y-intercept of (0, 1.414). This intercept can be found by evaluating *f(0)* like so: $f(0) = \sqrt{0 + 2} = \sqrt{2} = 1.414$



Red Line: $f(x) = \sqrt{x - 2}$

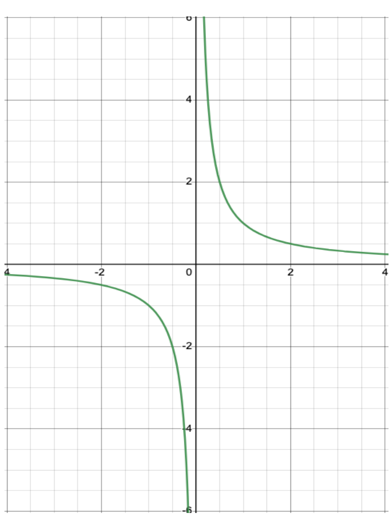
Domain: [2, ∞)
Range: [0, , ∞)

Blue Line: $f(x) = \sqrt{x + 2}$

Domain: [-2, ∞)
Range: [0, ∞)

Rational Functions
Now this is where functions get fun. Rational functions are basically composed of a ratio of two functions and typically follow the form $f(x) = \frac{A(x)}{B(x)}$. In this function, we can not let *B(x) = 0* because that will give us the form $\frac{1}{0}$ which means dividing by 0 which is not allowed in algebra. In fact, this is referred to as an *undefined* form. For example, consider the following rational function: $f(x) = \frac{1}{x}$

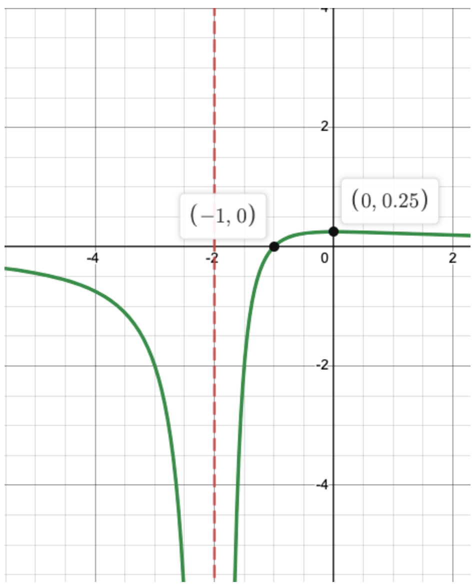
In this function, $x \neq 0$ but it can be any other value of x. This means that the domain of the $f(x) = \frac{1}{x}$ will be (0, ∞). However, we can also input negative numbers for x as well. So that means the domain of $f(x) = \frac{1}{x}$ is also (-∞, 0). To represent this, we use the symbol ∪ (union) to represent the full domain for $f(x) = \frac{1}{x}$ as (-∞, 0)∪(0, ∞). This function is represented by the green line below.



Since $x \neq 0$, this means that the green line gets closer and closer but never touches the y-axis. As such, the y-axis is referred to as the vertical asymptote for $f(x) = \frac{1}{x}$.

In a similar fashion, the x-axis is the horizontal asymptote for $f(x) = \frac{1}{x}$ since the green line gets closer and closer to the x-axis but never touches it as y approaches 0. So, this means that the range of $f(x) = \frac{1}{x}$ will be (-∞, 0)∪(0, ∞).

Example Problem: Find the asymptotes, intercepts, domain, and range for the function $f(x) = \frac{x+1}{(x+2)^2}$



Recall that the denominator (bottom of the fraction) can not be zero. So this means that $(x + 2)^2 \neq 0$. The only way that $(x + 2)^2 = 0$ would be if $x = -2$.

So this means that the line $x = -2$ will be the vertical asymptote for $f(x) = \frac{x+1}{(x+2)^2}$ This is represented by the dotted red line in the graphic. The green lines are the graph of $f(x) = \frac{x+1}{(x+2)^2}$

Domain: (-∞, -2)∪(-2, ∞)

Range: [0.25,-∞)

Finding the y-intercept when x = 0.

$$f(0) = \frac{0+1}{(0+2)^2} = \frac{1}{4} = 0.25$$

Therefore, the y-intercept is the point (0, 0.25).

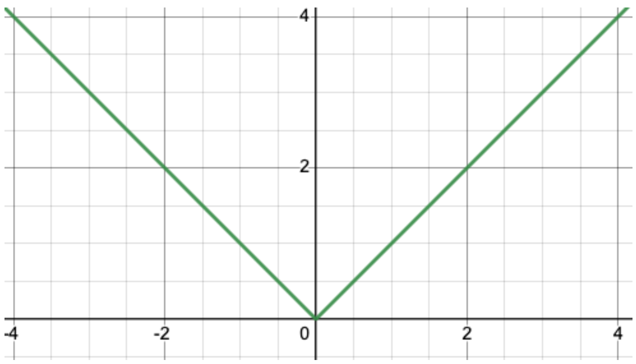
Finding the x-intercept when f(x) = 0.

$$0 = \frac{x+1}{(x+2)^2}$$

The only way this would equal 0 would be when x = -1. Recall that the numerator (top of fraction) must equal zero for the fraction to equal zero. Therefore, the x-intercept for this function will be (-1, 0).

Absolute Value Functions

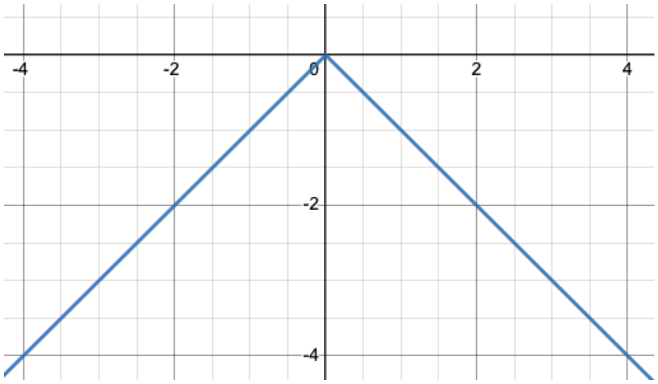
These functions will include the absolute value symbol and are of the form *f(x)* = |*x*|. Recall that any value x we place into |x| will be a positive result. For example both |2| and |-2| will equal positive 2. Similarly, both -|2| and -|-2| will equal -2. The graph below shows *f(x)* = |*x*|



For this function, the domain will be (-∞, ∞).

Notice that we are getting only positive y values here, so the range is [0, ∞).

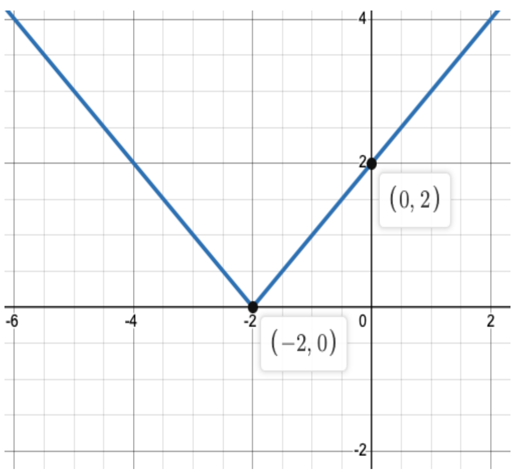
Here is the graph for $f(x) = -|x|$. The negative sign flips the line about the x-axis.



For this function, the domain will be $(-\infty, \infty)$.

Notice that we are now getting only negative y values here, so the range is $[0, -\infty)$.

Example Problem: Determine the intercepts, domain, and range for the function $f(x) = |x+2|$.

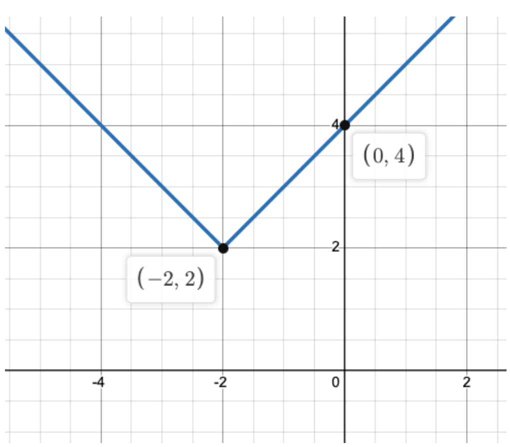


y-intercept: set $x=0$; $f(0) = |0+2| = 2$
(0,2) is the y-intercept

There is no x-intercept because $f(x) = |x+2|$ does not cross the x-axis. The point (-2, 0) is the vertex of $f(x) = |x+2|$.

For this function, the domain is $(-\infty, \infty)$ and the range is $[0, \infty)$.

Example Problem: Determine the intercepts, domain, and range for the function $f(x) = |x+2|+2$.



The y-intercept is the point (0, 4).

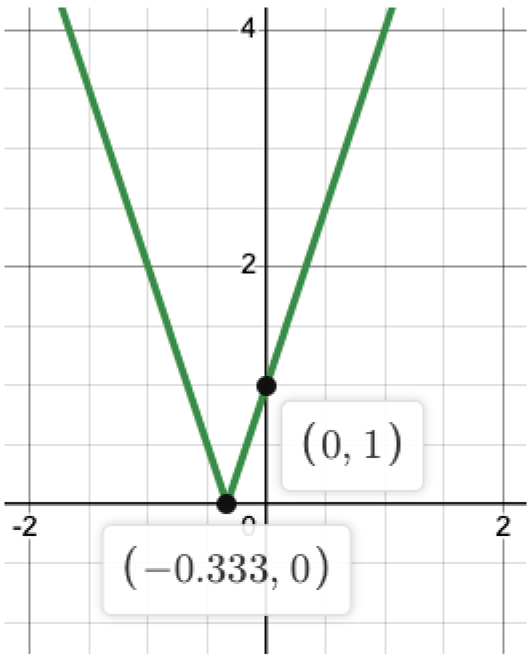
Similar to the last example, the function $f(x) = |x+2|+2$ does not cross the x-axis, so there is no x-intercept.

The vertex for the function $f(x) = |x+2|+2$ is the point (-2, 2).

By inspection, we can see that $|x+2|$ will be zero when $x=-2$. Evaluating at $x=-2$, we get $f(-2) = |-2+2|+2 = 4$. So the point (-2, 2) is the vertex.

Random Examples - Find the domain and range for each function. Identify any possible intercepts, asymptotes, or vertices then graph the function.

$f(x) = |3x+1|$



The vertex occurs where |3x+1|=0. That would occur when $x = -\frac{1}{3}$ (or -0.333) as shown below:

$f(-\frac{1}{3}) = |3(-\frac{1}{3})+1| = |-1+1| = 0$

The vertex is (-0.333,0). The y-intercept will occur when x=0.

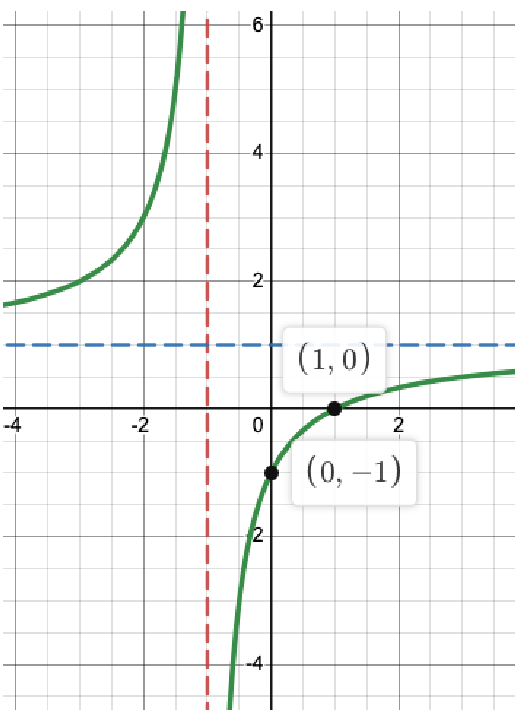
$f(0) = |3(0)+1| = 1$

Therefore, the y-intercept is the point (0, 1).

Domain: (-∞, ∞)

Range: [0, ∞).

$f(x) = \frac{x-1}{x+1}$



Since $x + 1 \neq 0$ because it would make the denominator 0, that means that the function has a vertical asymptote of x = -1 indicated by the red dotted line.

The horizontal asymptote is the line y = 1 indicated by the blue dotted line. The rules for finding the horizontal asymptote for rational functions are below.

y-intercept: $f(0) = \frac{0-1}{0+1} = -1$; the point (0, -1)

x-intercept: $f(x) = 0 = \frac{x-1}{x+1}$ this can only happen when x = 1, so the point (1, 0) is the x-intercept.

Domain: (-∞, -1)∪(-1, ∞)

Range: (-∞, 1)∪(1, ∞)

Horizontal Asymptote Rules for Rational Functions:

Given a rational function of the form $f(x) = \frac{x^n+a}{x^m+b}$, the three rules that horizontal asymptotes follow are based on the degree of the numerator, n, and the degree of the denominator, m.

If n < m, the horizontal asymptote is y = 0.

$f(x) = \frac{x-1}{x^2+2}$ Here n = 1 and m=2, so the horizontal asymptote will be y = 0.

If n = m, the horizontal asymptote is y = n/m.

$f(x) = \frac{x^2-1}{x^2+1}$ Here n = 2 and m = 2, so the horizontal asymptote will be y = 2/2 = 1

If n > m, there is no horizontal asymptote.

$f(x) = \frac{x^2-1}{x+1}$ Here n = 2 and m = 1, so there is no horizontal asymptote.