4- QUADRATIC FUNCTIONS

Now that you are familiar with different types of algebraic functions, let's take a closer look at the various approaches to solving quadratic functions. As shown previously, the graph of a quadratic function is simply a parabola whose solutions are related to whether or not the curve intercepts the x-axis. There are several different approaches to solving quadratic functions, but let's begin by taking a step back to the product of two binomial expressions. Consider the following example:

Example: What is the product of (x - 1) and (x + 2)?

To solve this problem, we are going to use a method known as FOIL. FOIL stands for **First, Outer, Inside** and **Last** and refers to the order in which the components are multiplied.

$(x - 1)(x + 2) = ?$

$(x - 1)(x + 2) = x^2 + \ldots$ *The **First** terms in both are x and x, so their product is x^2.*

$(x - 1)(x + 2) = x^2 + 2x + \ldots$ *The **Outer** terms are x and 2, so their product is 2x.*

$(x - 1)(x + 2) = x^2 + 2x - x + \ldots$ *The **Inside** terms are -1 and x, so their product is -x.*

$(x - 1)(x + 2) = x^2 + 2x - x - 2$ *The **Last** terms are -1 and 2, so their product is -2.*

$(x - 1)(x + 2) = x^2 + x - 2$ *Finally, add the middle terms 2x and -x, which gives x.*

Therefore, $(x - 1)(x + 2) = x^2 + x - 2$

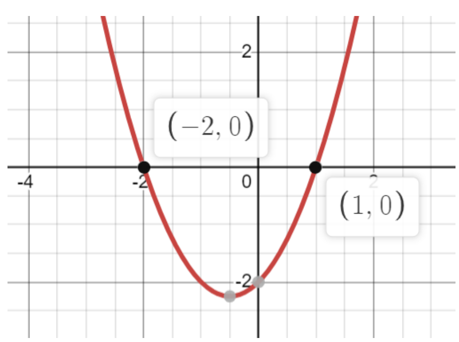
Try these examples on your own to make sure you know how to properly employ the FOIL method:

$$(x - 4)(x - 5) = x^2 - 9x + 20$$

$$(2x - 3)(3x + 2) = 6x^2 - 5x - 6$$

Find the Solutions to Quadratic Functions: Graphing and Factoring

Consider the quadratic function $f(x) = x^2 + x - 2$. The solutions to this function are the points on the x-axis where the parabola crosses. Basically, if there are any x-intercepts for the parabola, these are referred to as the solutions to the quadratic function when $f(x) = 0$. Below is the graph for $f(x) = x^2 + x - 2$. From this graph, we see that it crosses the x-axis at (-2, 0) and (1, 0).



Graphing is a quick method for determining the solutions (if they exist) to a quadratic formula. However, there are other methods for accomplishing this task. Recall that $(x - 1)(x + 2) = x^2 + x - 2$. If we set $(x - 1)(x + 2) = 0$, we see that this would only be possible when $x = 1$ or $x = -2$. Therefore, the points (1, 0) and (-2, 0) are solutions to the quadratic equation $x^2 + x - 2 = 0$. By factoring a quadratic equation, you can determine the solutions algebraically.

Example: What are the solutions to the quadratic equation $x^2 - 8x + 15 = 0$

There are many different approaches to factoring. One method is to ask yourself the following question:

What two numbers can I multiply together to get the last term that I can also add together to get the coefficient on the middle term?

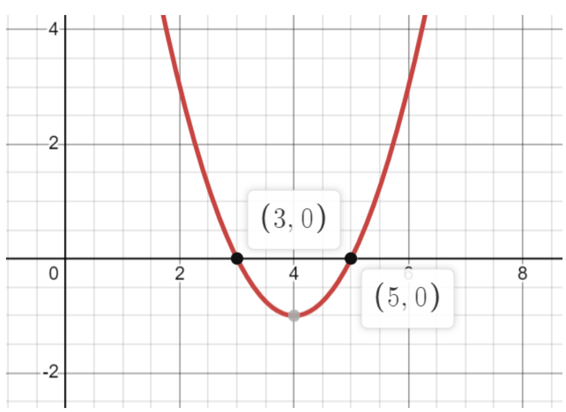
- Looking at $x^2 - 8x + 15 = 0$, we see that the last term is 15.
- We can multiply 3 and 5 to get 15.
- How can we add 3 and 5 to get -8?
- Well, to do that, they both have to be negative. So -3 added to -5 gives us -8.
- Therefore, $x^2 - 8x + 15$ factored is probably going to be $(x - 3)(x - 5)$.

Let's test this with FOIL.

$$(x - 3)(x - 5) = x^2 - 5x - 3x + 15 = x^2 - 8x + 15$$

Since we've successfully factored the quadratic, we can now find the solutions from $(x - 3)(x - 5) = 0$. This equation is only zero when $x = 3$ or $x = 5$. Therefore, the points (3, 0) and (5, 0) are the solutions to the quadratic equation $x^2 - 8x + 15 = 0$.

This is verified in the graph below of the function $f(x) = x^2 - 8x + 15$.



Example: What are the solutions to the quadratic equation $x^2 - 2x - 35 = 0$

Solution: We can multiply 5 and 7 to get 35. Notice that we have -35 in the equation, so either 5 or 7 must be negative. The clue to which is negative is revealed by the -2 coefficient on the middle term. To get that -2, it means that our 7 is probably going to be negative because 5 plus -7 gives us -2. So, our equation factored is most likely:

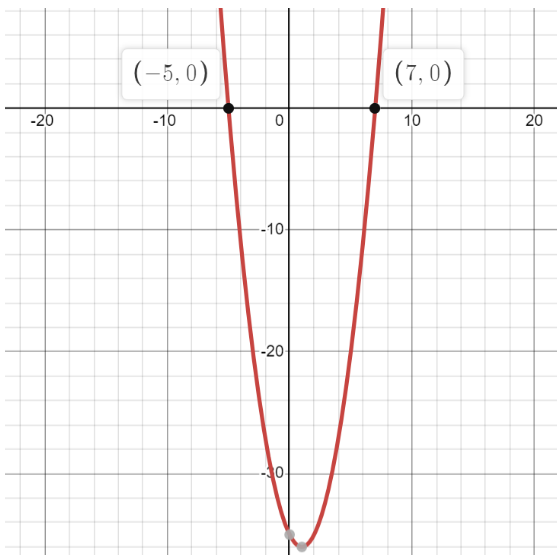
$$x^2 - 2x - 35 = (x + 5)(x - 7)$$

Once you get proficient at factoring quadratics, there is no need to verify with FOIL every time. However, let's go ahead and verify to make sure that our factoring is correct.

$$(x + 5)(x - 7) = x^2 - 7x + 5x - 35 = x^2 - 2x - 35$$

So, for $x^2 - 2x - 35 = (x + 5)(x - 7) = 0$, that means $x = -5$ or $x = 7$.

Therefore the solutions to $x^2 - 2x - 35 = 0$ are the points (-5, 0) and (7, 0) as shown in the graph below:



Find the Solutions to Quadratic Functions: Completing the Square

Suppose you are asked to find the solutions to the following equation: $x^2 - 12x + 7 = 0$. This particular equation doesn't lend itself to the factoring approach we just used because we can't readily come up with two numbers that multiplied will give 7 and added would give -12. So, we'll need to use another approach. Let's find the solutions to this quadratic equation using the approach called *completing the square.*

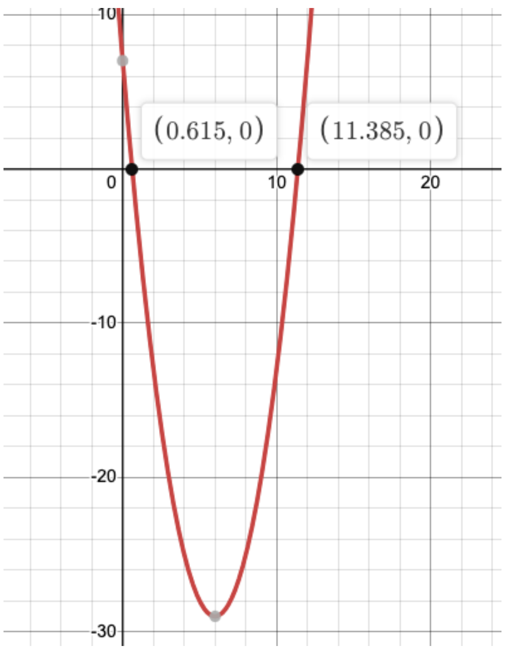
$x^2 - 12x + 7 = 0$	Let's start by subtracting 7 from both sides of the equation.
$x^2 - 12x = -7$	Now, let's try to get the left side into a perfect square of the form $(x + a)^2$.

Since $(x + a)^2 = x^2 + 2ax + a^2$, the middle coefficient will be given by 2a = -12. So, that means a = -6. It follows that $a^2 = (-6)^2 = 36$. So, we will add 36 to both sides of the equation.

$x^2 - 12x + 36 = -7 + 36$ Now, simplify to get …

$x^2 - 12x + 36 = 29$ The left side is now a perfect square of the form $(x + a)^2 = x^2 + 2ax + a^2$

$(x - 6)^2 = 29$ Therefore, $x^2 - 12x + 36 = (x - 6)^2$



Here's where it gets interesting. Take the square root of both sides to get the following:

$$x - 6 = \pm\sqrt{29}$$

Taking the square root of 29, means there are two possible roots: a positive and a negative value.

Now, add 6 to both sides to get our final solutions.

$$x = 6 \pm \sqrt{29}$$

So, the solutions to $x^2 - 12x + 7 = 0$ are the points $(6 + \sqrt{29}, 0)$ and $(6 - \sqrt{29}, 0)$ or (11.385, 0) and (0.615, 0). These solutions are verified by the graph of $f(x) = x^2 - 12x + 7$ on the left.

Example: Find the solutions to the quadratic functions $f(x) = x^2 - 6x + 1$ using the completing the square method.

$x^2 - 6x + 1 = 0$ Recall that the solutions are found when f(x) = 0.

$x^2 - 6x = -1$ Subtract 1 from both sides.

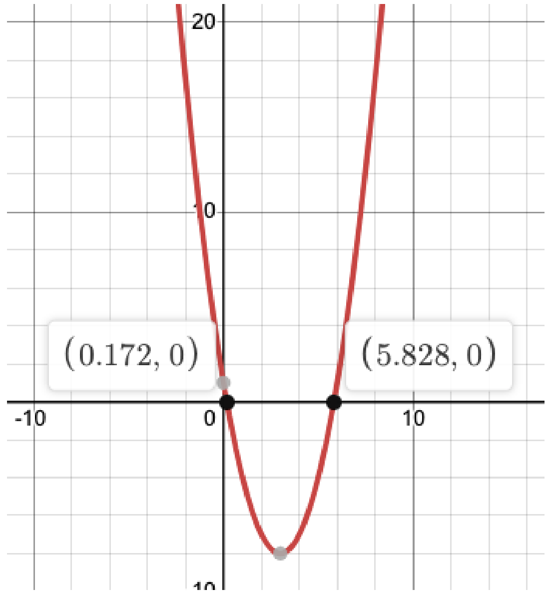
$x^2 - 6x + 9 = -1 + 9$ -6 divided by 2 is -3. The square of -3 is 9. Add 9 to both sides.

$(x - 3)^2 = 8$ $x^2 - 6x + 9 = (x - 3)^2$

$x - 3 = \pm\sqrt{8}$ Take the square root of both sides. Recall that we'll have 2 solutions on the right.

$x = 3 \pm \sqrt{8}$ Add 3 to both sides to get the final form of our solutions.

$x = 3 + \sqrt{8} = 5.828$ and $x = 3 - \sqrt{8} = 0.172$ are the possibilities here. So that means the points (0.172, 0) and (5.828, 0) are the solutions to the quadratic function $f(x) = x^2 - 6x + 1$ as shown below.



Find the Solutions to Quadratic Functions: The Dreaded Quadratic Formula
Let's be honest. The quadratic formula has served as algebra nightmare fuel for many centuries now. However, its negative reputation should be more attributed to the presentation of the concept rather than the utility of the formula. At a basic level, the quadratic formula is a clever tool that ancient mathematicians developed to find the solutions to quadratic functions that were not easily solved any other way.

It definitely isn't easy to remember, but we are not remembering anything for a test here anyway. You can think about the quadratic formula like you would a Phillips head screwdriver. You know what it looks like so you can find it in the tool box, right? The quadratic formula is like that, it's a screwdriver that, when you crank it, gives you the solutions to a quadratic equation.

Recall that the basic format for a quadratic equation is $ax^2 + bx + c = 0$. Where a, b, and c are coefficients and (*to make the ancient math people happy*) $a \neq 0$. Solving this basic form for x gives the famous quadratic formula (*I'll spare you the derivation*):

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example: Solve $-3x^2 + 4x + 6 = 0$ using the quadratic formula.

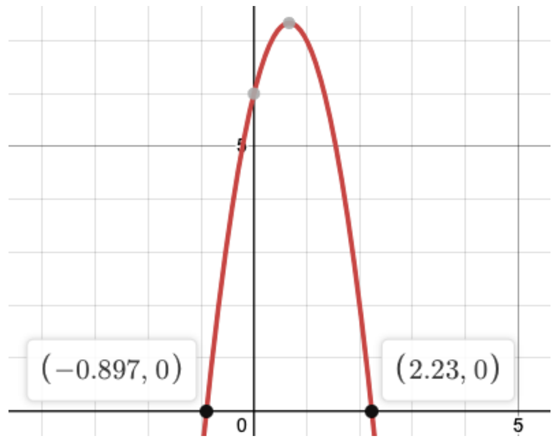
By inspection, we see that a = -3, b = 4, and c = 6. Plugging these into the quadratic formula gives:

$$x = \frac{-4 \pm \sqrt{4^2 - 4(-3)(6)}}{2(-3)} = \frac{-4 \pm \sqrt{88}}{-6} = \frac{-4 \pm 2\sqrt{22}}{-6} = \frac{2 \pm \sqrt{22}}{3}$$

So, the values for x that satisfy $-3x^2 + 4x + 6 = 0$ are

$$x = \frac{2+\sqrt{22}}{3} = 2.230 \text{ and } x = \frac{2-\sqrt{22}}{3} = -0.897$$

Therefore, the points (-0.897, 0) and (2.230, 0) are the solutions to $-3x^2 + 4x + 6 = 0$ as shown below.



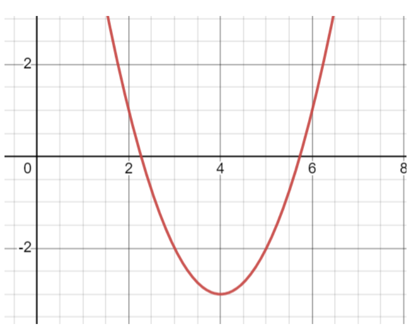
The Discriminant Reveals the Solutions!
Let's take another look at the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression $\boldsymbol{b^2 - 4ac}$ under the radical sign in the formula is called the **discriminant**. It has gained that name because it determines or discriminates the kinds of solutions possible for the quadratic equation in question. Here's how it works:

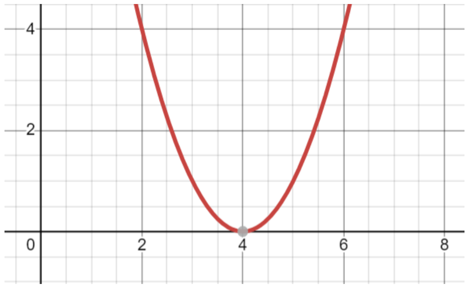
- If $\boldsymbol{b^2 - 4ac > 0,}$ that means there are going to be two unique real number solutions.
- If $\boldsymbol{b^2 - 4ac = 0,}$ that means there are going to be only one unique real number solutions.
- If $\boldsymbol{b^2 - 4ac < 0,}$ that means there are no real number solutions.

We can show this graphically as follows:



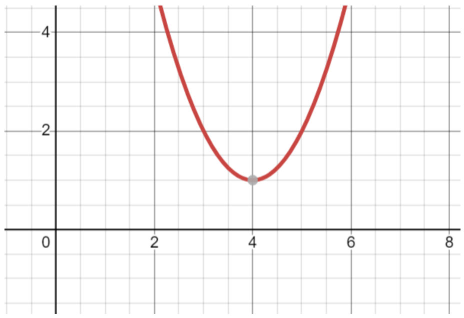
For this quadratic function, $\boldsymbol{b^2 - 4ac > 0,}$ meaning that there are two unique real number solutions.

Recall that the solutions to a quadratic function are the points where the parabola crosses the x-axis. This particular parabola crosses the x-axis at two points, hence two solutions.



For this quadratic function, b^2 *- 4ac = 0,* meaning that there is only one unique real number solution.

This particular parabola only touches the x-axis at one point, hence only one solution.



Finally, for this quadratic function, b^2 *- 4ac < 0,* meaning that there are no real number solutions.

There are no real solutions to this quadratic because it does not cross the x-axis at any point.

What does it mean to have no real solutions?

When the discriminant is b^2 *- 4ac < 0,* it means that you are getting a negative number under the radical sign in the quadratic formula. Recall that we have an item called the *imaginary number* which is defined as $i = \sqrt{-1}$. This will come into play in situations where the solutions to the quadratic are complex. Essentially, a complex number is composed of a real and an imaginary part. Despite the poor naming scheme provided to us by mathematicians long ago, there isn't anything imaginary about imaginary numbers. Complex numbers do in fact have many real world applications in science, engineering and business.

Example: Use the quadratic formula to find the solutions to $3x^2 + 5x + 12 = 0$.

Let's start by using the determinant to see if there are any real solutions. In this case, a = 3, b = 5, and c = 12. So, the determinant will be $b^2 - 4ac = (5)^2 - 4(3)(12) = 25 - 144 = -119$. Well, $-119 < 0$, so there are not going to be any real solutions to this quadratic equation, only complex ones! Let's find them now:

$$x = \frac{-5\pm\sqrt{5^2-4(3)(12)}}{2(3)} = \frac{-5\pm\sqrt{-119}}{6} = \frac{-5\pm\sqrt{119}i}{6}$$

Note: In the solutions above, $\sqrt{-119} = \sqrt{119} \times \sqrt{-1} = \sqrt{119}i$*, this means that the square root of -119 equals the square root of 119 multiplied by i.*

So, the solutions to $3x^2 + 5x + 12 = 0$ are when $x = \frac{-5-\sqrt{119}i}{6}$ and $x = \frac{-5+\sqrt{119}i}{6}$. Both of these are complex numbers. The real component is the $\frac{-5}{6}$ and the imaginary component is given by $\pm\frac{\sqrt{119}i}{6}$.

Application Area: The Profit Function

The profit of an operation is the result of the difference in revenue generated and the operational costs. If we represent Profit as P(x), Revenue as R(x) and Cost as C(x), the profit function can be obtained from P(x) = R(x) - C(x). When P(x) is represented as a quadratic function, we can model many aspects of the operation. For example, if we consider when P(x) = 0, we can determine if there are solutions that tell us the break even points of the operation. Let's do an example:

Example: Hysterical Orchards produces a wide range of apple-based products. The revenue for all apple-based products is given by the function R(x) = 266x and the total operational cost function for these same products is represented by $C(x) = 2x^2 + 46x + 2000$. What are the breakeven points for this operation?

First, let's determine what our P(x) function will look like:

$P(x) = R(x) - C(x) = 266x - (2x^2 + 46x + 2000) = 266x - 2x^2 - 46x - 2000 = -2x^2 + 220x - 2000$

Therefore, our profit function will be $P(x) = -2x^2 + 220x - 2000$

Break even occurs when P(x) = 0, so we'll set our quadratic function equal to 0.

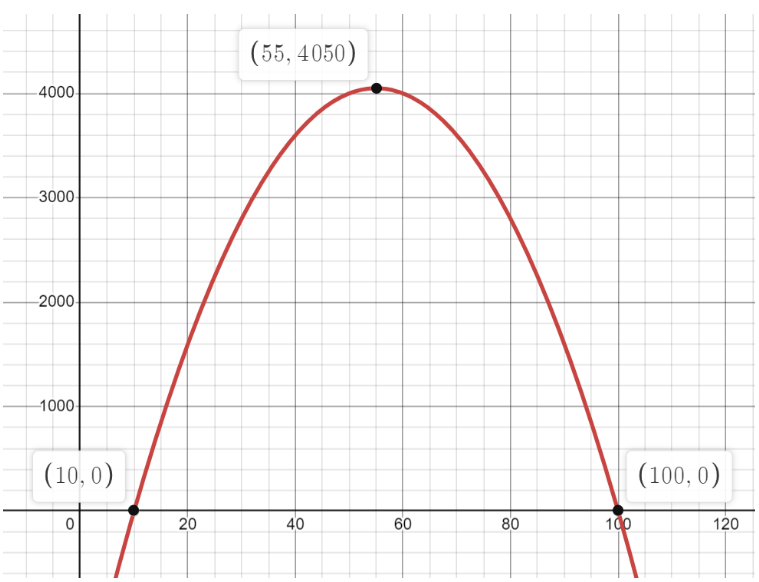
$-2x^2 + 220x - 2000 = 0$

Now, we can solve this using the quadratic formula.

$$x = \frac{-220 \pm \sqrt{220^2 - 4(-2)(-2000)}}{2(-2)} = \frac{-220 \pm \sqrt{32400}}{-4} = \frac{-220 \pm 180}{-4}$$

Therefore, the possible solutions are $x = \frac{40}{4} = 10$ and $x = \frac{400}{4} = 100$.

This means that Hysterical Orchards will break even at the point when it produces and sells 10 units or when it sells and produces 100 units.



Here's the graph of the profit function $P(x) = -2x^2 + 220x - 2000$.

On the graph, we see the break even points at (10, 0) and (100, 0).

Notice that the vertex of the parabola is the point (55, 4050). This point represents the point at which maximum profit is obtained given current cost and revenue functions.

To find the point of maximum profits, we need to find the vertex of the parabola. In general, for a quadratic function of the form $f(x) = ax^2 + bx + c$, the x coordinate of the vertex is given by the formula $x = \frac{-b}{2a}$ and the y coordinate is given by evaluating $f(\frac{-b}{2a})$.

Applying to our profit function $P(x) = -2x^2 + 220x - 2000$, we find the following:

$x = \frac{-b}{2a} = \frac{-220}{2(-2)} = 55$ $\quad f(55) = -2(55)^2 + 220(55) - 2000 = 4050$

Therefore, Hysterical Orchards achieves a maximum profit of $4050.00 when it produces and sells 55 units.

By the way, inputting the quadratic formula in calculations can be cumbersome. I highly recommend using the calculator at https://www.desmos.com/scientific.

There are also several quadratic formula calculators out there where all you need to do is input the coefficients and you'll get a complete solution. Here's one of my favorites at Calculator Soup: https://www.calculatorsoup.com/calculators/algebra/quadratic-formula-calculator.php

Finally, graphing quadratics will always be a good option for finding the solutions. Here's the online tool that was used to produce the graphs in this lecture: https://www.desmos.com/calculator