

5- EXPONENTIAL AND LOGARITHMIC FUNCTIONS

An exponential function is any mathematical function that can be expressed in the following form:

$$f(x) = ab^x$$

In this format, a is a constant and is typically the initial value of the function when $x = 0$. Recall that anything raised to the power of zero equals 1, so $f(0) = ab^0 = a(1) = a$. The value b is the base, and it is always greater than zero ($b > 0$). The variable is x and it represents the exponent of the function.

Example: Evaluate $f(x) = 2(10)^x$ when $x = 2$.

$$\text{Solution: } f(2) = 2(10)^2 = 2(100) = 200$$

Example: Evaluate $f(x) = 2(10)^{(x-1)}$ when $x = 2$.

$$\text{Solution: } f(2) = 2(10)^{(2-1)} = 2(10)^1 = 2(10) = 20$$

Exponential Growth

An exponential function can be used to model growth for a wide range of phenomena. In an exponential growth function, the base $(1+r)$ is always greater than 1 and $a > 0$. Exponential growth functions follow this format:

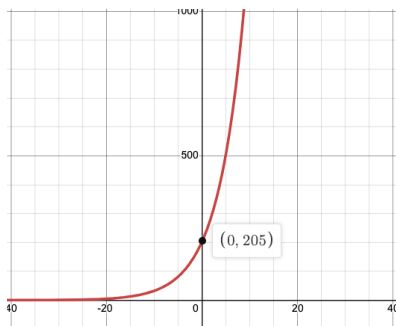
$$f(x) = a(1 + r)^x$$

In this formula, r represents the rate of growth.

Example: In 2020, the population of rabbits on Hop Island was estimated to be 205. If the annual rabbit population grows 20%, estimate the rabbit population on the island in 2030.

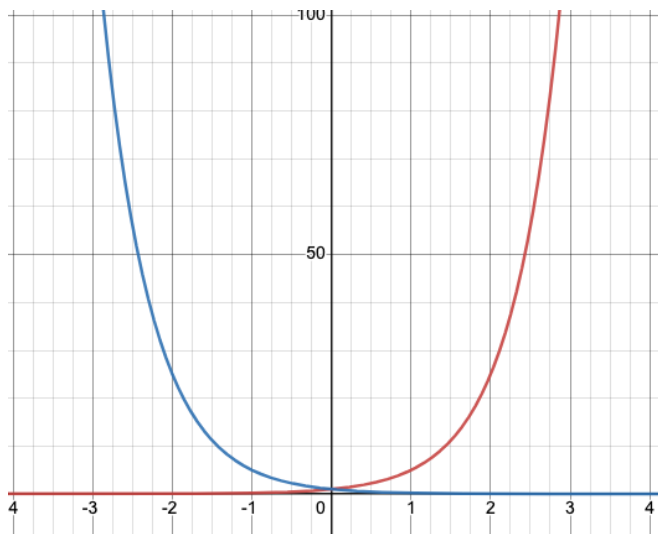
Solution: Our beginning population is $a = 205$, and our growth rate is 20% which will need to be expressed as a decimal in the formula so $r = 0.2$. Additionally, we are estimating the population after 10 years, so $x = 10$.

$$f(10) = 205(1 + 0.2)^{10} = 205(1.2)^{10} = 1269.3$$



We'll go ahead and round this result to 1269 because we don't want partial rabbits. Therefore, our estimated rabbit population on Hop Island in 2030 will be 1269.

The graph shows what $f(x) = 205(1.2)^x$ looks like. Notice that the y-intercept is the point $(0, 205)$ and that it is asymptotic with the x-axis.



The red line represents the function $f(x) = 5^x$ and the blue line represents the function $f(x) = 5^{-x}$.

Notice that the blue line is the reflection of the red line about the y-axis.

The red line represents an exponential growth function. The blue represents an exponential decay function.

Exponential Decay

An exponential function can be used to model decay for a wide range of phenomena. Exponential decay functions follow this format:

$$f(x) = a(1 - r)^x$$

In this exponential decay function, the base $(1-r)$ is always less than 1, $a > 0$, and r represents the rate of decay.

Another option is for the exponent to be negative. In that case, the exponential decay function will follow this format:

$$f(x) = ab^{-x}$$

In this version of the exponential decay function, $b > 1$ and $a > 0$.

Example: An adult Saguaro cactus can store up to 1500 gallons of water. During extreme dry periods, the Saguaro can lose up to $\frac{2}{3}$ of this stored water and still survive. The rate at which stored water is used depends upon a range of environmental conditions. If a particular adult Saguaro cactus uses 5% of its stored water monthly during an extremely long hot drought, estimate the amount of stored water that will remain after 6 months?

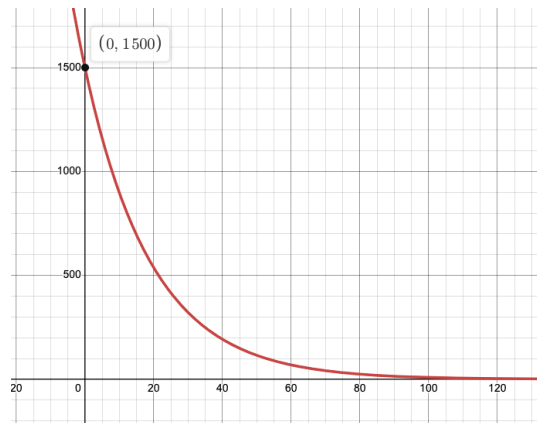
Solution: We need to set up our exponential decay function with the provided parameters. In this case, we know that our initial amount of water is 1500 gallons, so $a = 1500$. The rate at which water is used will be $r = 0.05$. Here's our exponential decay function:

$$f(x) = 1500(1 - 0.05)^x = 1500(0.95)^x$$

Now we are ready to estimate how much water will remain stored in the cactus after 6 months. To do this we evaluate $f(6)$.

$$f(6) = 1500(0.95)^6 = 1102.64 \text{ gallons}$$

Here's the graph of our exponential decay function $f(x) = 1500(0.95)^x$ which models water usage within an adult Saguaro cactus.

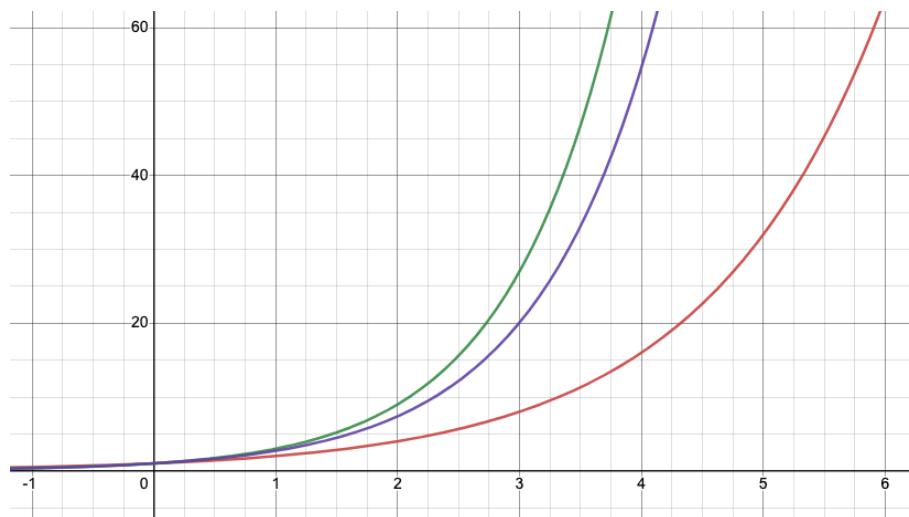


The Number e (Euler's Number)

There are many application areas in the sciences and business that are modeled by exponential functions with the base e . The number e is an irrational number that approximately equals:

$$e = 2.718281828\dots$$

In the graph below, the red line represents $f(x)=2^x$ and the green line represents $f(x)=3^x$. The graph of $f(x)=e^x$ is between these functions and is represented by the purple line.



A typical exponential function base e will follow this format: $f(x) = e^{rx}$

- If $r > 1$, then the function base e will represent exponential growth.
- If $0 < r < 1$, then the function base e will represent exponential decay.

Example: Evaluate the function $f(x) = 2e^{5x}$ when $x = 2$.

Solution: $f(x) = 2e^{5x} = 2e^{5(2)} = 2e^{10} = 44052.93$

It is highly recommended that you use a scientific calculator for these computations. Here's what this solution would look like using <https://www.desmos.com/scientific>.



The typical format for an exponential decay formula base e will be $f(x) = ae^{-rx}$

For example, an exponential decay function is used to model the amount of radioactive material remaining after a period of time. Carbon-14 decays over time with the amount remaining after t years represented by the following function:

$$f(t) = y_0 e^{-0.000121t}$$

Where y_0 is the initial amount of carbon-14 material and $r = 0.000121$ is the rate of change per year.

Example: If the original amount of carbon-14 present in an artifact is 100 grams, how much remains after 2000 years?

Solution: $f(2000) = 100e^{-0.000121(2000)} = 78.51 \text{ grams}$

Here's the solution using <https://www.desmos.com/scientific>.



Let's do another one.

Example: What percentage of the original amount of carbon-14 remains after 4500 years?

Solution: We know that y_o is the initial amount of carbon-14 material, so plugging $t = 4500$ into our formula gives $f(4500) = y_o e^{-0.000121(4500)} = 0.57y_o$.

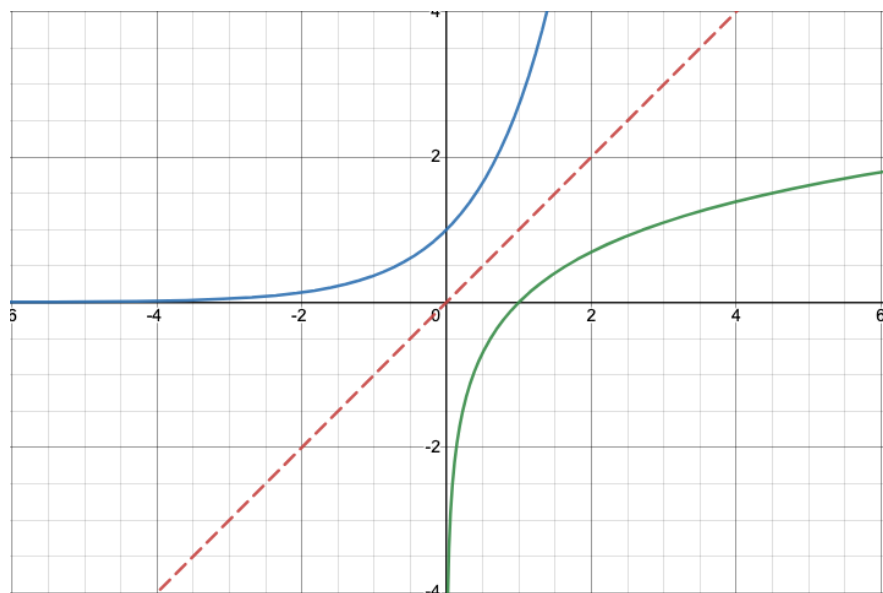
Therefore, 57% of the carbon-14 will remain after 4500 years.

Incidentally, the half-life for carbon-14 is 5730 years which means that after this time only 50% of the original carbon-14 remains.

Logarithmic Functions

Logarithmic functions have the following format: $f(x) = \log_b x$

Logarithmic functions are the inverse of exponential functions. In general, a function is represented as $y = f(x)$ and its inverse function is represented as $x = f(y)$. It is important to note that a function and its inverse function must have a one-to-one correspondence between the independent and dependent variables defining the function. The inverse function for a logarithmic function is an exponential function. This is shown graphically below:



The green line is the graph of $f(x) = \ln x$. This function is called the *natural log of x* and it is base e .

The blue line is the function $f(x) = e^x$ which is the inverse function for $f(x) = \ln x$.

Since $f(x) = \ln x$ and $f(x) = e^x$ are inverses of each other, their graphs are symmetrical about the line $y = x$ represented by the red dotted line.

Therefore, since logarithmic functions have the format $y = \log_b x$, the corresponding exponential form will be $x = b^y$, and its inverse will have the format $y = b^x$.

These are referred to as the logarithmic forms and exponential forms respectively as demonstrated by the examples in the following table:

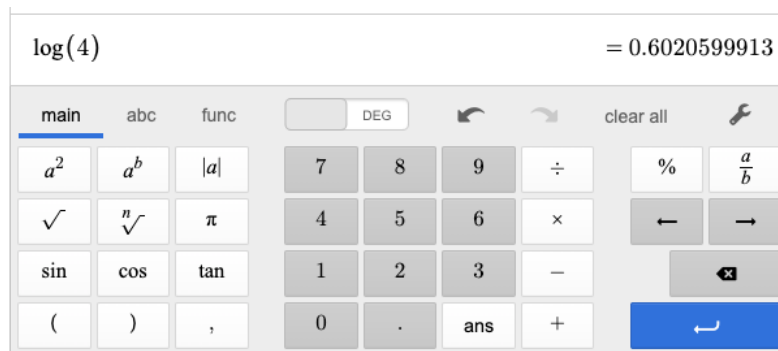
Logarithmic Form	Exponential Form
$y = \log_2 x$	$x = 2^y$
$2 = \log_{10} 100$	$10^2 = 100$
$2 = \log_3 9$	$3^2 = 9$

Example: Evaluate the function $f(x) = \log_{10} x$ when $x = 4$.

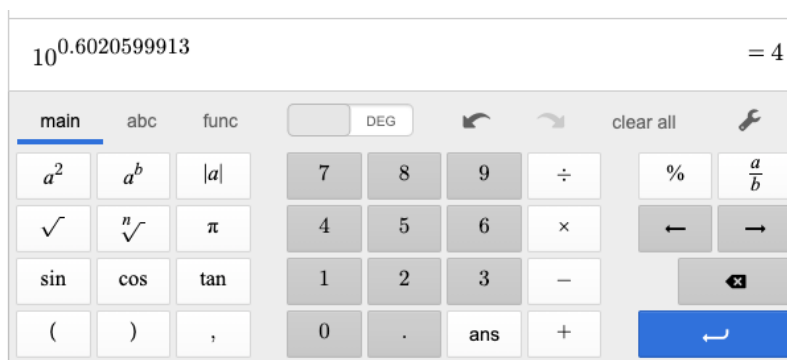
To find $f(4)$, we can input this into a calculator to quickly find the solution. Scientific calculators come with a log function button that is base 10 by default.

$$f(4) = \log_{10} 4 = 0.6020599913$$

Here's the solution using <https://www.desmos.com/scientific>. The log button is found on the **func** menu.



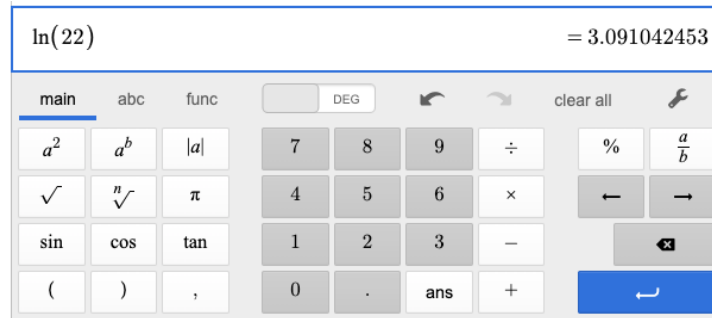
So, our logarithmic form works out to be $\log_{10} 4 = 0.6020599913$. The exponential form for this would be $10^{0.6020599913} = 4$ which we can verify with the calculator as shown below:



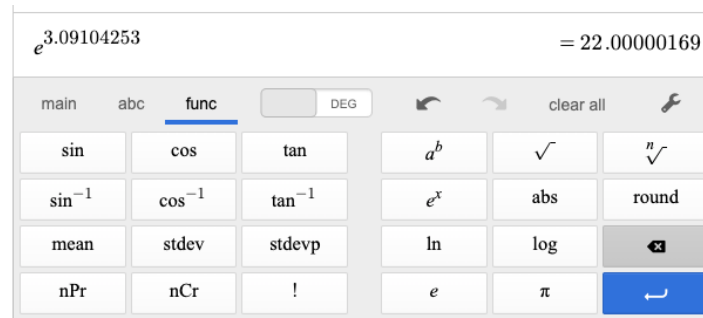
Example: Evaluate the function $f(x) = \ln x$ when $x = 22$.

Another common logarithm is the *natural logarithm* which is a logarithm based e . This is defined as follows: $\log_e x = \ln x$. It is handy to know that all scientific calculators come with an \ln function button.

So, let's use <https://www.desmos.com/scientific> to evaluate $f(22) = \ln 22$. Here's the solution:



Therefore, $f(22) = \ln 22 = 3.09104253$. The exponential form would be $e^{3.09104253} = 22$ as verified below:



An interesting property of logarithmic functions is the following:

$$f(x) = \log x^n = n \log x$$

This is called the **power property** of logarithms. Let's put this property to use in the following example:

Example: Evaluate the function $f(x) = \log x^4$ when $x = 10$. Round your answer to the nearest hundredths.

Solution: $f(10) = \log 10^4 = 4 \log 10 = 4(1) = 4$

This solution employed the following basic property of logarithms: $\log_b b = 1$

The following list contains some of the basic logarithmic properties (for $b > 0$, $b \neq 1$):

- $\log_b b = 1$
- $\log_b 1 = 0$
- $\log_b b^x = x$
- $b^{\log x} = x$

The following list contains some useful operational properties of logarithms (for $b > 0$, $b \neq 1$).

- Product Property: $\log (MN) = \log M + \log N$
- Quotient Property: $\log \left(\frac{M}{N}\right) = \log M - \log N$
- Product Property: $\log M^n = n \log M$

A Logarithmic Function Application: pH

The acidity or basicity of a solution depends upon the concentration of free hydrogen ions $[H^+]$ in the solution. The pH scale is a logarithmic scale that represents a simple way to represent the hydrogen ion concentration of a solution. The pH of a solution is given by the following formula:

$$\text{pH} = -\log[H^+]$$

Example: Find the pH value of beer for which the hydrogen concentration is $[H^+] = 0.0000631$.

Solution: $\text{pH} = -\log(0.0000631) = 4.199970641$. Therefore, the pH of this beer is 4.2.

Example: The pH value of eggs is 7.79. Find the hydrogen ion concentration for eggs.

Solution: Since the logarithmic form is $7.79 = -\log[H^+]$, it follows that the exponential form will give the solution $[H^+] = 10^{-7.79} = 0.00000001628101$