2-SYSTEMS OF LINEAR EQUATIONS

There are many systems in the world around us that can be modeled with linear equations. Some of these systems may require more than one linear equation to properly model the behavior of the system or allow for comparison to other similar systems. In these scenarios, the linear equations that model different systems will have the same variables making it possible to determine if there is some point of intersection that both systems have in common.

Consider the system of linear equations composed of the following lines:

$$y = 3x - 7$$
$$y = 5x - 8$$

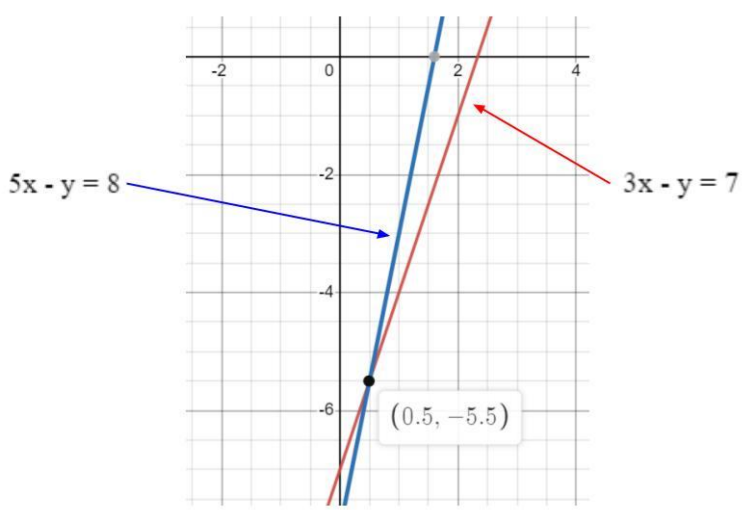
Here we have two different linear equations both modeling the behavior of some system. These equations can also be represented as follows:

$$3x - y = 7$$
$$5x - y = 8$$

The solution to this system of linear equations is the point where the two lines intersect. In this lecture, we are going to cover three methods for determining the solution to a system of linear equations: graphing, substitution, and elimination. For this course, we are going to stick to systems of linear equations that only have two variables. However, you should be aware that the methods we cover here can be applied to more complex linear systems with more than two variables. When there are 3 or more variables, typically the system of linear equations is solved using matrix algebra which will not be covered in this course.

Solving a System of Linear Equations by Graphing

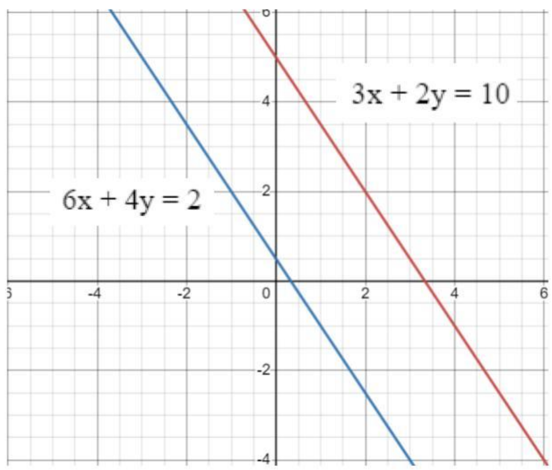
This is the most straightforward approach to solving a system of linear equations because it involves drawing each line and determining where they intersect from the graph. In the past, this would have been achieved by finding the x and y intercepts for both lines, then drawing the lines on graph paper. However, as a student in the 21st century, you have many tools available to you to accomplish this task quickly. Here is the graph of our system of linear equations using the online graphing calculator https://www.desmos.com/calculator.



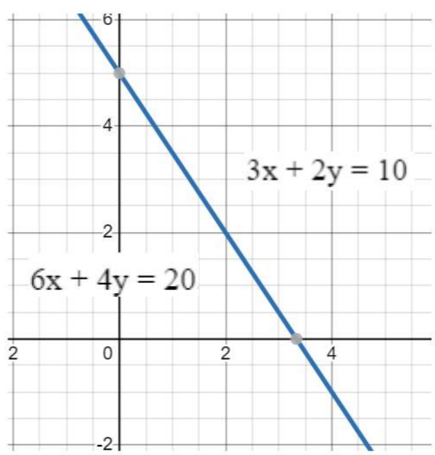
From graphing the system represented by 3x - y = 7 and 5x - y = 8, we find that these two lines intersect at the point (0.5, -5.5). **This point represents the solution to this system of linear equations.** This is an example of a system of linear equations that actually has a solution because the lines have a definite intersection. Consider the following system:

$$3x + 2y = 10$$
$$6x + 4y = 2$$

Graphing this system, we see that the lines do not intersect because they are parallel to each other. Therefore this is an example of a system of linear equations that **does not have a solution**.



Take a look at the following graph of the linear system composed of 3x + 2y = 10 and 6x + 4y = 20.



A quick inspection of 6x + 4y = 20 reveals that if you divide by 2, you would get 3x + 2y = 10. So, these lines are really the exact same line. As such, they share an infinite number of points. **This is interpreted as a system that has an infinite number of solutions.**

Now, let's consider two ways that you can solve a system of linear equations algebraically.

Solving a System of Linear Equations by Substitution
As the name implies, we are going to solve one equation for one variable then substitute it into the other equation to find the value of the second variable. Once we have that value, we insert it back into one of the equations to find the value of the initial variable.

For example, let's find the solution to the system of linear equations represented by the following:

$$x + 3y = 1$$
$$2x - 3y = 2$$

We'll start with the first equation and solve it for x.

$$x + 3y = 1$$
$$x = 1 - 3y$$

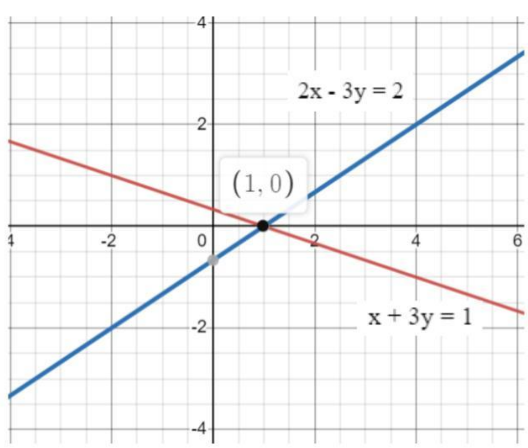
Now, we'll substitute x = 1 - 3y into the equation 2x - 3y = 2 and solve for y.

$$2x - 3y = 2$$
$$2(1 - 3y) - 3y = 2$$
$$2 - 6y - 3y = 2$$
$$2 - 9y = 2$$
$$-9y = 2 - 2$$
$$-9y = 0$$
$$y = 0$$

Next, we'll substitute y = 0 into one of our original equations (it doesn't matter which one you pick).

$$x + 3y = 1$$
$$x + 3(0) = 1$$
$$x = 1$$

Therefore, the point (1, 0) is the solution to this system of linear equations as shown in the graph below:



Solving a System of Linear Equations by Elimination

This method is also called solving a system of simultaneous equations by elimination. It provides the framework for the algebra involved in solving matrices of linear equations with 3 or more variables. In this approach, we will use the coefficients in the equations to essentially eliminate one of the variables so we can find the value for the other. Once we have that value, we can insert it into one of the equations to solve for the other variable value.

Let's consider the following linear equation:

$$4x + y = 5$$
$$8x + 3y = 2$$

Notice that the coefficient on the y variable in the equation $4x + y = 5$ is 1. This makes this equation the perfect candidate to manipulate so we can eliminate the y variable. Notice too that the coefficient on the y variable in the equation $8x + 3y = 2$ is a positive 3. If we multiply $4x + y = 5$ by a negative 3, we can add the equations together to eliminate the y variable as follows:

Multiply by $4x + y = 5$ by -3:

$$-3\{4x + y = 5\}$$
$$8x + 3y = 2$$

This results in the following:

$$-12x - 3y = -15$$
$$8x + 3y = 2$$

Now, add the column together as follows:

$$-12x - 3y = -15$$
$$\underline{8x + 3y = 2}$$
$$-4x + 0 = -13$$

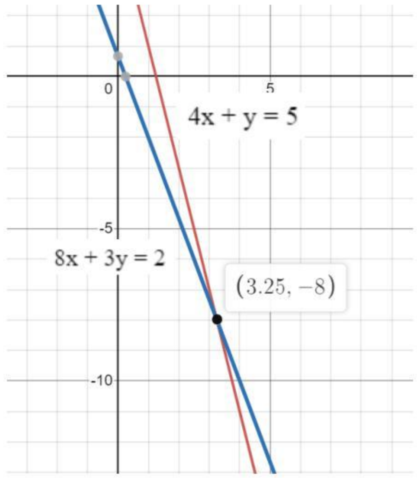
Next, we solve for x:

$$-4x = -13$$
$$x = 13/4 = 3.25$$

Insert $x = 3.25$ into one of the equations (doesn't matter which):

$$4x + y = 5$$
$$4(3.25) + y = 5$$
$$13 + y = 5$$
$$y = -8$$

Therefore, the point (3.25, -8) is the solution to this system of linear equations. We can verify that by graphing the lines as shown below.



Example 1
The honey production (in pounds) of Dikembe's apiary is represented by the linear equation $x + 2y = 320$. In this equation, x represents the number of wildflower acres near his apiary where the bees harvest pollen and y represents the number of hives. Liu operates an apiary in a neighboring county where the operation has been modeled with the linear equation $x - 8y = 20$. Find the wildflower acreage and number of hives where both apiaries would expect to see the same level of honey produced.

To find the answer, we will solve the system of linear equations: $x + 2y = 320$ and $x - 8y = 20$. Let's do this via the substitution method.

Using $x - 8y = 20$, we see that when we solve for x, we get the following: $x = 8y + 20$

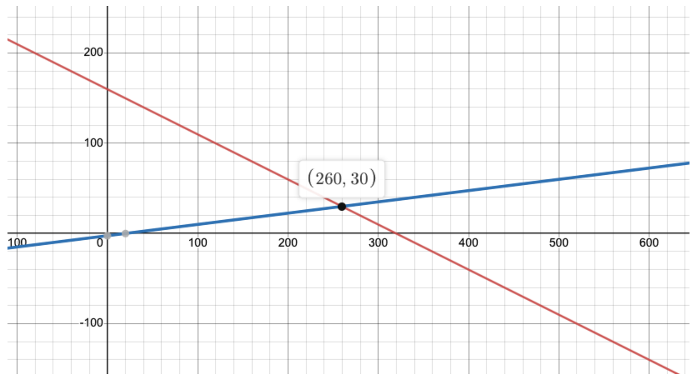
Substituting $x = 8y + 20$ into $x + 2y = 320$ to solve for y.

$$x + 2y = 320$$
$$(8y + 20) + 2y = 320$$
$$10y + 20 = 320$$
$$10y = 300$$
$$y = 30$$

Now, insert $y = 30$ into one of the linear equations.

$$x + 2y = 320$$
$$x + 2(30) = 320$$
$$x + 60 = 320$$
$$x = 260$$

Therefore, the solution to this system of linear equations is (260, 30). This means that both apiaries will have the same honey production if their bees have access to 260 acres of wildflowers and the number of hives is 30. Here's the graphical solution:



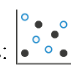
Example 2

A small agroforestry operation integrates growing soybeans within pecan tree rows in a method referred to as alley cropping. The table below tracks the operational costs and revenue generated by the soybean operation as a function of units of soybean sold. Using this information, determine the breakeven point for the soybean operation.

Soybean Units (x)	Revenue (y)	Operational Costs (y)
10	841.5	3564
20	1545.3	3545.1
30	2340.9	3855.6
40	3243.6	4293
50	3901.5	4406.4
60	5049	5049
70	5729.85	5200.2
80	6609.6	5540.4
90	7504.65	5886
100	7879.5	5840.1
110	9004.05	6355.8
120	9822.6	6644.7
130	10740.6	6998.4
140	11245.5	7087.5
150	12622.5	7722
160	12362.4	7362.9
170	14045.4	8164.8
180	14596.2	8299.8
190	14971.05	8343

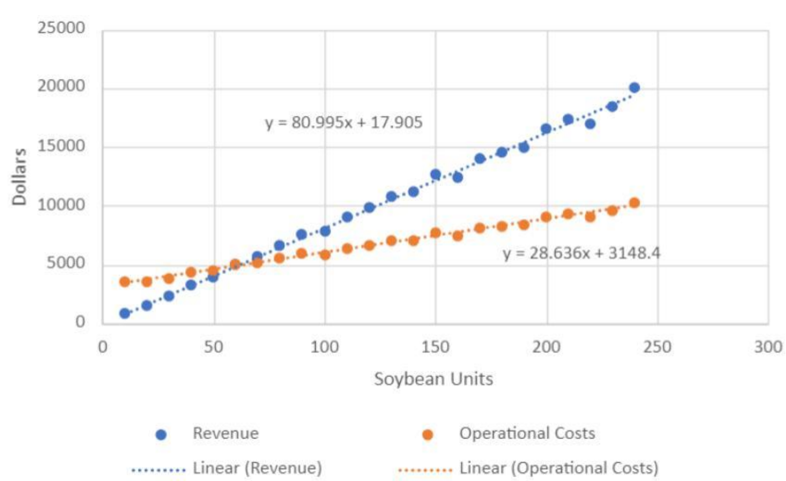
200	16524	9039.6
210	17350.2	9331.2
220	16998.3	8999.1
230	18474.75	9639
240	20012.4	10300.5

To tackle this problem, let's begin by importing our data into a spreadsheet program such as Excel or Google Sheets. We want to determine the linear equation for the trendlines that represent soybean revenue and soybean operational costs. The solution to this system of linear equations will represent the breakeven point for the soybean operation. Here are the instructions for accomplishing this task in Excel (*web-based version included with Office 365*).

- Copy and paste data into a new sheet.
- Select the data columns.
- On the menu bar, click Insert, then Chart.
- From the chart menu options, select the one for scatterplot. It should look like this: 
- Label the scatter plot.
  - Right-click on the chart and then select format.
  - Remove the chart title that auto populated. Keep the legend which indicates which color dots correspond to Revenue and Operational Costs.
  - Label the horizontal axis by finding the option for *Axis Title* under the *Horizontal Axis* menu. Label the horizontal axis as Soybean Units**.**
  - Label the vertical axis by finding the option for *Axis Title* under the *Vertical Axis* menu. The vertical axis should be labeled as Dollars**.**
- Create a trendline through the Revenue data points on your scatter plot and display the model equation.
  - From the chart format menu, under the *Series "Revenue"* menu, activate the *Trendline* option.
  - Make sure that the *Trend Type* is *Linear.*
  - Select the *Display Equation on Chart* option.

- Create a trendline through the Operational Costs data points on your scatter plot and display the model equation.
  - From the chart format menu, under the *Series "Operational Costs"* menu, activate the *Trendline* option.
  - Make sure that the *Trend Type* is *Linear.*
  - Select the *Display Equation on Chart* option.

Here's the completed chart.



From this chart, we see that the linear equation for operational costs is y = 28.636x + 3148.4 and the linear equation for revenue is y = 80.995x + 17.905.

This system is already set up nicely to be solved using substitution. Substituting the second equation for the value of y in the first equation we get the equation below. Let's solve it for x.

$$80.995x + 17.905 = 28.636x + 3148.4$$
$$80.995x - 28.636x = 3148.4 - 17.905$$
$$52.359x = 3130.495$$
$$x = 59.789$$

Now that we have a value for x, let's find y.

$$y = 28.636(59.789) + 3148.4 = 4860.519$$

Therefore, the solution to our system of linear equations is (59.789, 4860.519). This means that the soybean operation's break even point is $4860.519 when 59.789 units are produced. We can verify this solution by graphing the equations then finding the intersection point as shown below.